This huge crank belongs to a Wartsila-Sulzer RTA96-C turbocharged two-stroke diesel engine. In this chapter you will learn to perform the kinematic analysis of rigid bodies that undergo translation, fixed axis rotation, and general plane motion.
CHAPTER

Kinematics of Rigid Bodies
15.1 INTRODUCTION

In this chapter, the kinematics of rigid bodies will be considered. You will investigate the relations existing between the time, the positions, the velocities, and the accelerations of the various particles forming a rigid body. As you will see, the various types of rigid-body motion can be conveniently grouped as follows:

1. Translation. A motion is said to be a translation if any straight line inside the body keeps the same direction during the motion. It can also be observed that in a translation all the particles forming the body move along parallel paths. If these paths are straight lines, the motion is said to be a rectilinear translation (Fig. 15.1); if the paths are curved lines, the motion is a curvilinear translation (Fig. 15.2).

2. Rotation about a Fixed Axis. In this motion, the particles forming the rigid body move in parallel planes along circles centered on the same fixed axis (Fig. 15.3). If this axis, called the axis of rotation, intersects the rigid body, the particles located on the axis have zero velocity and zero acceleration.

Rotation should not be confused with certain types of curvilinear translation. For example, the plate shown in Fig. 15.4a is in curvilinear translation, with all its particles moving along parallel circles, while the plate shown in Fig. 15.4b is in rotation, with all its particles moving along concentric circles.
In the first case, any given straight line drawn on the plate will maintain the same direction, whereas in the second case, point $O$ remains fixed.

Because each particle moves in a given plane, the rotation of a body about a fixed axis is said to be a plane motion.

3. General Plane Motion. There are many other types of plane motion, i.e., motions in which all the particles of the body move in parallel planes. Any plane motion which is neither a rotation nor a translation is referred to as a general plane motion. Two examples of general plane motion are given in Fig. 15.5.

4. Motion about a Fixed Point. The three-dimensional motion of a rigid body attached at a fixed point $O$, e.g., the motion of a top on a rough floor (Fig. 15.6), is known as motion about a fixed point.

5. General Motion. Any motion of a rigid body which does not fall in any of the categories above is referred to as a general motion.

After a brief discussion in Sec. 15.2 of the motion of translation, the rotation of a rigid body about a fixed axis is considered in Sec. 15.3. The angular velocity and the angular acceleration of a rigid body about a fixed axis will be defined, and you will learn to express the velocity and the acceleration of a given point of the body in terms of its position vector and the angular velocity and angular acceleration of the body.

The following sections are devoted to the study of the general plane motion of a rigid body and to its application to the analysis of mechanisms such as gears, connecting rods, and pin-connected linkages. Resolving the plane motion of a slab into a translation and a rotation (Secs. 15.5 and 15.6), we will then express the velocity of a point $B$ of the slab as the sum of the velocity of a reference point $A$ and of the velocity of $B$ relative to a frame of reference translating with $A$ (i.e., moving with $A$ but not rotating). The same approach is used later in Sec. 15.8 to express the acceleration of $B$ in terms of the acceleration of $A$ and of the acceleration of $B$ relative to a frame translating with $A$. 

\textbf{Fig. 15.5}

(a) Rolling wheel

(b) Sliding rod

\textbf{Fig. 15.6}

$O$
An alternative method for the analysis of velocities in plane motion, based on the concept of *instantaneous center of rotation*, is given in Sec. 15.7; and still another method of analysis, based on the use of parametric expressions for the coordinates of a given point, is presented in Sec. 15.9.

The motion of a particle relative to a rotating frame of reference and the concept of *Coriolis acceleration* are discussed in Secs. 15.10 and 15.11, and the results obtained are applied to the analysis of the plane motion of mechanisms containing parts which slide on each other.

The remaining part of the chapter is devoted to the analysis of the three-dimensional motion of a rigid body, namely, the motion of a rigid body with a fixed point and the general motion of a rigid body. In Secs. 15.12 and 15.13, a fixed frame of reference or a frame of reference in translation will be used to carry out this analysis; in Secs. 15.14 and 15.15, the motion of the body relative to a rotating frame or to a frame in general motion will be considered, and the concept of Coriolis acceleration will again be used.

### 15.2 TRANSLATION

Consider a rigid body in translation (either rectilinear or curvilinear translation), and let $A$ and $B$ be any two of its particles (Fig. 15.7a). Denoting, respectively, by $r_A$ and $r_B$ the position vectors of $A$ and $B$ with respect to a fixed frame of reference and by $r_{BA}$ the vector joining $A$ and $B$, we write

$$r_B = r_A + r_{BA} \quad (15.1)$$

Let us differentiate this relation with respect to $t$. We note that from the very definition of a translation, the vector $r_{BA}$ must maintain a constant direction; its magnitude must also be constant, since $A$ and $B$
belong to the same rigid body. Thus, the derivative of $r_{BA}$ is zero and we have

$$v_B = v_A \quad (15.2)$$

Differentiating once more, we write

$$a_B = a_A \quad (15.3)$$

Thus, when a rigid body is in translation, all the points of the body have the same velocity and the same acceleration at any given instant (Fig. 15.7b and c). In the case of curvilinear translation, the velocity and acceleration change in direction as well as in magnitude at every instant. In the case of rectilinear translation, all particles of the body move along parallel straight lines, and their velocity and acceleration keep the same direction during the entire motion.

### 15.3 Rotation about a Fixed Axis

Consider a rigid body which rotates about a fixed axis $AA'$. Let $P$ be a point of the body and $r$ its position vector with respect to a fixed frame of reference. For convenience, let us assume that the frame is centered at point $O$ on $AA'$ and that the $z$ axis coincides with $AA'$ (Fig. 15.8). Let $B$ be the projection of $P$ on $AA'$; since $P$ must remain at a constant distance from $B$, it will describe a circle of center $B$ and of radius $r \sin \phi$, where $\phi$ denotes the angle formed by $r$ and $AA'$.

The position of $P$ and of the entire body is completely defined by the angle $\theta$ the line $BP$ forms with the $zx$ plane. The angle $\theta$ is known as the angular coordinate of the body and is defined as positive when viewed as counterclockwise from $A'$. The angular coordinate will be expressed in radians (rad) or, occasionally, in degrees (°) or revolutions (rev). We recall that

$$1 \text{ rev} = 2\pi \text{ rad} = 360°$$

We recall from Sec. 11.9 that the velocity $v = dr/dt$ of a particle $P$ is a vector tangent to the path of $P$ and of magnitude $v = ds/dt$. Observing that the length $\Delta s$ of the arc described by $P$ when the body rotates through $\Delta \theta$ is

$$\Delta s = (BP) \Delta \theta = (r \sin \phi) \Delta \theta$$

and dividing both members by $\Delta t$, we obtain at the limit, as $\Delta t$ approaches zero,

$$v = \frac{ds}{dt} = r \dot{\theta} \sin \phi \quad (15.4)$$

where $\dot{\theta}$ denotes the time derivative of $\theta$. (Note that the angle $\theta$ depends on the position of $P$ within the body, but the rate of change $\dot{\theta}$ is itself independent of $P$.) We conclude that the velocity $v$ of $P$ is a vector perpendicular to the plane containing $AA'$ and $r$, and of

**Fig. 15.8**

**Photo 15.2** For the central gear rotating about a fixed axis, the angular velocity and angular acceleration of that gear are vectors directed along the vertical axis of rotation.
magnitude \( v \) defined by (15.4). But this is precisely the result we would obtain if we drew along \( \mathbf{AA}' \) a vector \( \mathbf{\omega} = \dot{\theta} \mathbf{k} \) and formed the vector product \( \mathbf{\omega} \times \mathbf{r} \) (Fig. 15.9). We thus write

\[
\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{\omega} \times \mathbf{r} \tag{15.5}
\]

The vector

\[
\mathbf{\omega} = \omega \mathbf{k} = \dot{\theta} \mathbf{k} \tag{15.6}
\]

which is directed along the axis of rotation, is called the angular velocity of the body and is equal in magnitude to the rate of change \( \dot{\theta} \) of the angular coordinate; its sense may be obtained by the right-hand rule (Sec. 3.6) from the sense of rotation of the body.†

The acceleration \( \mathbf{a} \) of the particle \( P \) will now be determined. Differentiating (15.5) and recalling the rule for the differentiation of a vector product (Sec. 11.10), we write

\[
\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\mathbf{\omega} \times \mathbf{r})
= \frac{d\mathbf{\omega}}{dt} \times \mathbf{r} + \mathbf{\omega} \times \frac{d\mathbf{r}}{dt}
= \frac{d\mathbf{\omega}}{dt} \times \mathbf{r} + \mathbf{\omega} \times \mathbf{v} \tag{15.7}
\]

The vector \( d\mathbf{\omega}/dt \) is denoted by \( \mathbf{\alpha} \) and is called the angular acceleration of the body. Substituting also for \( \mathbf{v} \) from (15.5), we have

\[
\mathbf{a} = \mathbf{\alpha} \times \mathbf{r} + \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) \tag{15.8}
\]

Differentiating (15.6) and recalling that \( \mathbf{k} \) is constant in magnitude and direction, we have

\[
\mathbf{\alpha} = \alpha \mathbf{k} = \ddot{\omega} \mathbf{k} = \dot{\theta} \mathbf{k} \tag{15.9}
\]

Thus, the angular acceleration of a body rotating about a fixed axis is a vector directed along the axis of rotation, and is equal in magnitude to the rate of change \( \dot{\omega} \) of the angular velocity. Returning to (15.8), we note that the acceleration of \( P \) is the sum of two vectors. The first vector is equal to the vector product \( \mathbf{\alpha} \times \mathbf{r} \); it is tangent to the circle described by \( P \) and therefore represents the tangential component of the acceleration. The second vector is equal to the vector triple product \( \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) \) obtained by forming the vector product of \( \mathbf{\omega} \) and \( \mathbf{\omega} \times \mathbf{r} \); since \( \mathbf{\omega} \times \mathbf{r} \) is tangent to the circle described by \( P \), the vector triple product is directed toward the center \( B \) of the circle and therefore represents the normal component of the acceleration.

†It will be shown in Sec. 15.12 in the more general case of a rigid body rotating simultaneously about axes having different directions that angular velocities obey the parallelogram law of addition and thus are actually vector quantities.
**Rotation of a Representative Slab.** The rotation of a rigid body about a fixed axis can be defined by the motion of a representative slab in a reference plane perpendicular to the axis of rotation. Let us choose the $xy$ plane as the reference plane and assume that it coincides with the plane of the figure, with the $z$ axis pointing out of the paper (Fig. 15.10). Recalling from (15.6) that $\mathbf{\omega} = \omega \mathbf{k}$, we note that a positive value of the scalar $\omega$ corresponds to a counterclockwise rotation of the representative slab, and a negative value to a clockwise rotation. Substituting $\omega \mathbf{k}$ for $\mathbf{\omega}$ into Eq. (15.5), we express the velocity of any given point $P$ of the slab as

$$\mathbf{v} = \omega \mathbf{k} \times \mathbf{r}$$  \hspace{1cm} (15.10)

Since the vectors $\mathbf{k}$ and $\mathbf{r}$ are mutually perpendicular, the magnitude of the velocity $\mathbf{v}$ is

$$v = r \omega$$  \hspace{1cm} (15.10')

and its direction can be obtained by rotating $\mathbf{r}$ through $90^\circ$ in the sense of rotation of the slab.

Substituting $\mathbf{\omega} = \omega \mathbf{k}$ and $\mathbf{\alpha} = \alpha \mathbf{k}$ into Eq. (15.8), and observing that cross-multiplying $\mathbf{r}$ twice by $\mathbf{k}$ results in a $180^\circ$ rotation of the vector $\mathbf{r}$, we express the acceleration of point $P$ as

$$\mathbf{a} = \alpha \mathbf{k} \times \mathbf{r} - \omega^2 \mathbf{r}$$  \hspace{1cm} (15.11)

Resolving $\mathbf{a}$ into tangential and normal components (Fig. 15.11), we write

$$\mathbf{a}_t = \alpha \mathbf{k} \times \mathbf{r} \quad a_t = r \alpha$$  \hspace{1cm} (15.11')

$$\mathbf{a}_n = -\omega^2 \mathbf{r} \quad a_n = r \omega^2$$

The tangential component $\mathbf{a}_t$ points in the counterclockwise direction if the scalar $\alpha$ is positive, and in the clockwise direction if $\alpha$ is negative. The normal component $\mathbf{a}_n$ always points in the direction opposite to that of $\mathbf{r}$, that is, toward $O$. 
15.4 EQUATIONS DEFINING THE ROTATION OF A RIGID BODY ABOUT A FIXED AXIS

The motion of a rigid body rotating about a fixed axis $\mathbf{A}\mathbf{A}'$ is said to be known when its angular coordinate $\theta$ can be expressed as a known function of $t$. In practice, however, the rotation of a rigid body is seldom defined by a relation between $\theta$ and $t$. More often, the conditions of motion will be specified by the type of angular acceleration that the body possesses. For example, $\alpha$ may be given as a function of $t$, as a function of $\theta$, or as a function of $\omega$. Recalling the relations (15.6) and (15.9), we write

$$\omega = \frac{d\theta}{dt}$$

(15.12)

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

(15.13)

or, solving (15.12) for $dt$ and substituting into (15.13),

$$\alpha = \omega \frac{d\omega}{d\theta}$$

(15.14)

Since these equations are similar to those obtained in Chap. 11 for the rectilinear motion of a particle, their integration can be performed by following the procedure outlined in Sec. 11.3.

Two particular cases of rotation are frequently encountered:

1. Uniform Rotation. This case is characterized by the fact that the angular acceleration is zero. The angular velocity is thus constant, and the angular coordinate is given by the formula

$$\theta = \theta_0 + \omega t$$

(15.15)

2. Uniformly Accelerated Rotation. In this case, the angular acceleration is constant. The following formulas relating angular velocity, angular coordinate, and time can then be derived in a manner similar to that described in Sec. 11.5. The similarity between the formulas derived here and those obtained for the rectilinear uniformly accelerated motion of a particle is apparent.

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

(15.16)

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

It should be emphasized that formula (15.15) can be used only when $\alpha = 0$, and formulas (15.16) can be used only when $\alpha = \text{constant}$. In any other case, the general formulas (15.12) to (15.14) should be used.
SAMPLE PROBLEM 15.1

Load B is connected to a double pulley by one of the two inextensible cables shown. The motion of the pulley is controlled by cable C, which has a constant acceleration of 9 in./s² and an initial velocity of 12 in./s, both directed to the right. Determine (a) the number of revolutions executed by the pulley in 2 s, (b) the velocity and change in position of the load B after 2 s, and (c) the acceleration of point D on the rim of the inner pulley at \( t = 0 \).

SOLUTION

a. Motion of Pulley. Since the cable is inextensible, the velocity of point D is equal to the velocity of point C and the tangential component of the acceleration of D is equal to the acceleration of C.

\[
(v_D)_0 = (v_C)_0 = 12 \text{ in./s} \quad \Rightarrow \quad (a_D)_t = a_C = 9 \text{ in./s}^2
\]

Noting that the distance from D to the center of the pulley is 3 in., we write

\[
(v_D)_0 = r\omega_0 = 12 \text{ in./s} = (3 \text{ in.})\omega_0 \quad \Rightarrow \quad \omega_0 = 4 \text{ rad/s}
\]

\[
(a_D)_t = r\alpha = 9 \text{ in./s}^2 = (3 \text{ in.})\alpha \quad \Rightarrow \quad \alpha = 3 \text{ rad/s}^2
\]

Using the equations of uniformly accelerated motion, we obtain, for \( t = 2 \text{ s} \),

\[
\omega = \omega_0 + \alpha t = 4 \text{ rad/s} + (3 \text{ rad/s}^2)(2 \text{ s}) = 10 \text{ rad/s}
\]

\[
\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = (4 \text{ rad/s})(2 \text{ s}) + \frac{1}{2}(3 \text{ rad/s}^2)(2 \text{ s})^2 = 14 \text{ rad}
\]

Number of revolutions = \((14 \text{ rad})\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right) = 2.23 \text{ rev}\)

b. Motion of Load B. Using the following relations between linear and angular motion, with \( r = 5 \text{ in.} \), we write

\[
v_B = r\omega = (5 \text{ in.})(10 \text{ rad/s}) = 50 \text{ in./s}
\]

\[
\Delta y_B = r\theta = (5 \text{ in.})(14 \text{ rad}) = 70 \text{ in.}
\]

\[
v_B = 50 \text{ in./s} \uparrow \quad \Delta y_B = 70 \text{ in. upward}
\]

c. Acceleration of Point D at \( t = 0 \). The tangential component of the acceleration is

\[
(a_D)_t = a_C = 9 \text{ in./s}^2
\]

Since, at \( t = 0 \), \( \omega_0 = 4 \text{ rad/s} \), the normal component of the acceleration is

\[
(a_D)_n = r\omega_0^2 = (3 \text{ in.})(4 \text{ rad/s})^2 = 48 \text{ in./s}^2 \quad (a_D)_n = 48 \text{ in./s}^2 \downarrow
\]

The magnitude and direction of the total acceleration can be obtained by writing

\[
\tan \phi = (48 \text{ in./s}^2)/(9 \text{ in./s}^2) \quad \Rightarrow \quad \phi = 79.4^\circ
\]

\[
a_D \sin 79.4^\circ = 48 \text{ in./s}^2 \quad \Rightarrow \quad a_D = 48.8 \text{ in./s}^2 \nabla 79.4^\circ
\]
SOLVING PROBLEMS ON YOUR OWN

In this lesson we began the study of the motion of rigid bodies by considering two particular types of motion of rigid bodies: translation and rotation about a fixed axis.

1. Rigid body in translation. At any given instant, all the points of a rigid body in translation have the same velocity and the same acceleration (Fig. 15.7).

2. Rigid body rotating about a fixed axis. The position of a rigid body rotating about a fixed axis was defined at any given instant by the angular coordinate $\theta$, which is usually measured in radians. Selecting the unit vector $\mathbf{k}$ along the fixed axis and in such a way that the rotation of the body appears counterclockwise as seen from the tip of $\mathbf{k}$, we defined the angular velocity $\omega$ and the angular acceleration $\alpha$ of the body:

$$\omega = \dot{\theta} \mathbf{k} \quad \alpha = \ddot{\theta} \mathbf{k} \quad (15.6, 15.9)$$

In solving problems, keep in mind that the vectors $\omega$ and $\alpha$ are both directed along the fixed axis of rotation and that their sense can be obtained by the right-hand rule.

a. The velocity of a point $P$ of a body rotating about a fixed axis was found to be

$$v = \omega \times r \quad (15.5)$$

where $\omega$ is the angular velocity of the body and $r$ is the position vector drawn from any point on the axis of rotation to point $P$ (Fig. 15.9).

b. The acceleration of point $P$ was found to be

$$a = \alpha \times r + \omega \times (\omega \times r) \quad (15.8)$$

Since vector products are not commutative, be sure to write the vectors in the order shown when using either of the above two equations.

3. Rotation of a representative slab. In many problems, you will be able to reduce the analysis of the rotation of a three-dimensional body about a fixed axis to the study of the rotation of a representative slab in a plane perpendicular to the fixed axis. The $z$ axis should be directed along the axis of rotation and point out of the paper. Thus, the representative slab will be rotating in the $xy$ plane about the origin $O$ of the coordinate system (Fig. 15.10).

To solve problems of this type you should do the following:

a. Draw a diagram of the representative slab, showing its dimensions, its angular velocity and angular acceleration, as well as the vectors representing the velocities and accelerations of the points of the slab for which you have or seek information.
b. Relate the rotation of the slab and the motion of points of the slab by writing the equations

\[ v = r \omega \]  \hspace{1cm} (15.10')
\[ a_t = r \alpha \quad a_n = r \omega^2 \]  \hspace{1cm} (15.11')

Remember that the velocity \( v \) and the component \( a_t \) of the acceleration of a point \( P \) of the slab are tangent to the circular path described by \( P \). The directions of \( v \) and \( a_t \) are found by rotating the position vector \( r \) through 90° in the sense indicated by \( \omega \) and \( \alpha \), respectively. The normal component \( a_n \) of the acceleration of \( P \) is always directed toward the axis of rotation.

4. Equations defining the rotation of a rigid body. You must have been pleased to note the similarity existing between the equations defining the rotation of a rigid body about a fixed axis [Eqs. (15.12) through (15.16)] and those in Chap. 11 defining the rectilinear motion of a particle [Eqs. (11.1) through (11.8)]. All you have to do to obtain the new set of equations is to substitute \( \theta \), \( \omega \), and \( \alpha \) for \( x \), \( v \), and \( a \) in the equations of Chap. 11.
15.1 The motion of a cam is defined by the relation \( \theta = t^3 - 9t^2 + 15t \), where \( \theta \) is expressed in radians and \( t \) in seconds. Determine the angular coordinate, the angular velocity, and the angular acceleration of the cam when (a) \( t = 0 \), (b) \( t = 3 \) s.

15.2 For the cam of Prob. 15.1, determine the time, angular coordinate, and angular acceleration when the angular velocity is zero.

15.3 The motion of an oscillating crank is defined by the relation
\[
\theta = \theta_0 \sin \left( \frac{\pi t}{T} \right) - (0.5\theta_0) \sin \left( 2 \frac{\pi t}{T} \right)
\]
where \( \theta \) is expressed in radians and \( t \) in seconds. Knowing that \( \theta_0 = 6 \) rad and \( T = 4 \) s, determine the angular coordinate, the angular velocity, and the angular acceleration of the crank when (a) \( t = 0 \), (b) \( t = 2 \) s.

15.4 Solve Prob. 15.4, when \( t = 1 \) s.

15.5 The motion of a disk rotating in an oil bath is defined by the relation
\[
\theta = \theta_0 \left( 1 - e^{-t/\tau} \right)
\]
where \( \theta \) is expressed in radians and \( t \) in seconds. Knowing that \( \theta_0 = 0.40 \) rad, determine the angular coordinate, velocity, and acceleration of the disk when (a) \( t = 0 \), (b) \( t = 3 \) s, (c) \( t = \infty \).

15.6 The angular acceleration of an oscillating disk is defined by the relation
\[
\alpha = -k\theta
\]
Determine (a) the value of \( k \) for which \( \omega = 8 \) rad/s when \( \theta = 0 \) and \( \theta = 4 \) rad when \( \omega = 0 \), (b) the angular velocity of the disk when \( \theta = 3 \) rad.

15.7 When the power to an electric motor is turned on the motor reaches its rated speed of 3300 rpm in 6 s, and when the power is turned off the motor coasts to rest in 80 s. Assuming uniformly accelerated motion, determine the number of revolutions that the motor executes (a) in reaching its rated speed, (b) in coasting to rest.

15.8 The rotor of a gas turbine is rotating at a speed of 6900 rpm when the turbine is shut down. It is observed that 4 min is required for the rotor to coast to rest. Assuming uniformly accelerated motion, determine (a) the angular acceleration, (b) the number of revolutions that the rotor executes before coming to rest.

15.9 The angular acceleration of a shaft is defined by the relation
\[
\alpha = -0.25\omega, \quad \text{where} \quad \alpha \text{ is expressed in rad/s}^2 \text{ and} \ \omega \text{ in rad/s.}
\]
Knowing that at \( t = 0 \) the angular velocity of the shaft is 20 rad/s, determine (a) the number of revolutions the shaft will execute before coming to rest, (b) the time required for the shaft to come to rest, (c) the time required for the angular velocity of the shaft to be reduced to 1 percent of its initial value.
15.10 The assembly shown consists of the straight rod $ABC$ which passes through and is welded to the rectangular plate $DEFH$. The assembly rotates about the axis $AC$ with a constant angular velocity of 9 rad/s. Knowing that the motion when viewed from $C$ is counterclockwise, determine the velocity and acceleration of corner $F$.

![Diagram of the assembly](Fig. P15.10)

15.11 In Prob. 15.10, determine the acceleration of corner $H$, assuming that the angular velocity is 9 rad/s and decreases at a rate of 18 rad/s².

15.12 The bent rod $ABCDE$ rotates about a line joining points $A$ and $E$ with a constant angular velocity of 9 rad/s. Knowing that the rotation is clockwise as viewed from $E$, determine the velocity and acceleration of corner $C$.

![Diagram of the bent rod](Fig. P15.12)

15.13 In Prob. 15.12, determine the velocity and acceleration of corner $B$, assuming that the angular velocity is 9 rad/s and increases at the rate of 45 rad/s².
15.14 A triangular plate and two rectangular plates are welded to each other and to the straight rod AB. The entire welded unit rotates about axis AB with a constant angular velocity of \(5\) rad/s. Knowing that at the instant considered the velocity of corner E is directed downward, determine the velocity and acceleration of corner D.

![Fig. P15.14](image_url)

15.15 In Prob. 15.14, determine the acceleration of corner D, assuming that the angular velocity is \(5\) rad/s and decreases at the rate of \(20\) rad/s\(^2\).

15.16 The earth makes one complete revolution on its axis in 23 h 56 min. Knowing that the mean radius of the earth is 3960 mi, determine the linear velocity and acceleration of a point on the surface of the earth (a) at the equator, (b) at Philadelphia, latitude 40° north, (c) at the North Pole.

15.17 The earth makes one complete revolution around the sun in 365.24 days. Assuming that the orbit of the earth is circular and has a radius of 93,000,000 mi, determine the velocity and acceleration of the earth.

15.18 The circular plate shown is initially at rest. Knowing that \(r = 200\) mm and that the plate has a constant angular acceleration of 0.3 rad/s\(^2\), determine the magnitude of the total acceleration of point B when (a) \(t = 0\), (b) \(t = 2\) s, (c) \(t = 4\) s.

15.19 The angular acceleration of the 600-mm-radius circular plate shown is defined by the relation \(\alpha = \alpha_0 e^{-t}\). Knowing that the plate is at rest when \(t = 0\) and that \(\alpha_0 = 10\) rad/s\(^2\), determine the magnitude of the total acceleration of point B when (a) \(t = 0\), (b) \(t = 0.5\) s, (c) \(t = \infty\).

15.20 The 250-mm-radius circular plate shown is initially at rest and has an angular acceleration defined by the relation \(\alpha = \alpha_0 \cos (\pi t/T)\). Knowing that \(T = 1.5\) s and \(\alpha_0 = 10\) rad/s\(^2\), determine the magnitude of the total acceleration of point B when (a) \(t = 0\), (b) \(t = 0.5\) s, (c) \(t = 0.75\) s.
15.21 A series of small machine components being moved by a conveyor belt pass over a 6-in.-radius idler pulley. At the instant shown, the velocity of point $A$ is 15 in./s to the left and its acceleration is 9 in./s$^2$ to the right. Determine (a) the angular velocity and angular acceleration of the idler pulley, (b) the total acceleration of the machine component at $B$.

![Fig. P15.21 and P15.22](image)

15.22 A series of small machine components being moved by a conveyor belt pass over a 6-in.-radius idler pulley. At the instant shown, the angular velocity of the idler pulley is 4 rad/s clockwise. Determine the angular acceleration of the pulley for which the magnitude of the total acceleration of the machine component at $B$ is 120 in./s$^2$.

15.23 The belt sander shown is initially at rest. If the driving drum $B$ has a constant angular acceleration of 120 rad/s$^2$ counterclockwise, determine the magnitude of the acceleration of the belt at point $C$ when (a) $t = 0.5$ s, (b) $t = 2$ s.

![Fig. P15.23 and P15.24](image)

15.24 The rated speed of drum $B$ of the belt sander shown is 2400 rpm. When the power is turned off, it is observed that the sander coasts from its rated speed to rest in 10 s. Assuming uniformly decelerated motion, determine the velocity and acceleration of point $C$ of the belt, (a) immediately before the power is turned off, (b) 9 s later.
15.25 Ring C has an inside radius of 55 mm and an outside radius of 60 mm and is positioned between two wheels A and B, each of 24-mm outside radius. Knowing that wheel A rotates with a constant angular velocity of 300 rpm and that no slipping occurs, determine (a) the angular velocity of the ring C and of wheel B, (b) the acceleration of the points A and B which are in contact with C.

15.26 Ring B has an inside radius $r_2$ and hangs from the horizontal shaft A as shown. Knowing that shaft A rotates with a constant angular velocity $\omega_A$ and that no slipping occurs, derive a relation in terms of $r_1$, $r_2$, $r_3$, and $\omega_A$ for (a) the angular velocity of ring B, (b) the accelerations of the points of shaft A and ring B which are in contact.

15.27 Ring B has an inside radius $r_2$ and hangs from the horizontal shaft A as shown. Shaft A rotates with a constant angular velocity of 25 rad/s and no slipping occurs. Knowing that $r_1 = 12$ mm, $r_2 = 30$ mm, and $r_3 = 40$ mm, determine (a) the angular velocity of ring B, (b) the accelerations of the points of shaft A and ring B which are in contact, (c) the magnitude of the acceleration of a point on the outside surface of ring B.

15.28 Cylinder A is moving downward with a velocity of 9 ft/s when the brake is suddenly applied to the drum. Knowing that the cylinder moves 18 ft downward before coming to rest and assuming uniformly accelerated motion, determine (a) the angular acceleration of the drum, (b) the time required for the cylinder to come to rest.

15.29 The system shown is held at rest by the brake-and-drum system. After the brake is partially released at $t = 0$, it is observed that the cylinder moves 16 ft in 5 s. Assuming uniformly accelerated motion, determine (a) the angular acceleration of the drum, (b) the angular velocity of the drum at $t = 4$ s.
15.30 A pulley and two loads are connected by inextensible cords as shown. Load A has a constant acceleration of 300 \( \text{mm/s}^2 \) and an initial velocity of 240 \( \text{mm/s} \), both directed upward. Determine 
(a) the number of revolutions executed by the pulley in 3 s, 
(b) the velocity and position of load B after 3 s, 
(c) the acceleration of point D on the rim of the pulley at \( t = 0 \).

15.31 A pulley and two loads are connected by inextensible cords as shown. The pulley starts from rest at \( t = 0 \) and is accelerated at the uniform rate of 2.4 \( \text{rad/s}^2 \) clockwise. At \( t = 4 \) s, determine the velocity and position 
(a) of load A, 
(b) of load B.

15.32 Disk B is at rest when it is brought into contact with disk A which is rotating freely at 450 rpm clockwise. After 6 s of slippage, during which each disk has a constant angular acceleration, disk A reaches a final angular velocity of 140 rpm clockwise. Determine the angular acceleration of each disk during the period of slippage.

15.33 and 15.34 A simple friction drive consists of two disks A and B. Initially, disk A has a clockwise angular velocity of 500 rpm and disk B is at rest. It is known that disk A will coast to rest in 60 s. However, rather than waiting until both disks are at rest to bring them together, disk B is given a constant angular acceleration of 2.5 \( \text{rad/s}^2 \) counterclockwise. Determine 
(a) at what time the disks can be brought together if they are not to slip, 
(b) the angular velocity of each disk as contact is made.

15.35 Two friction disks A and B are both rotating freely at 240 rpm counterclockwise when they are brought into contact. After 8 s of slippage, during which each disk has a constant angular acceleration, disk A reaches a final angular velocity of 60 rpm counterclockwise. Determine 
(a) the angular acceleration of each disk during the period of slippage, 
(b) the time at which the angular velocity of disk B is equal to zero.
15.36 In a continuous printing process, paper is drawn into the presses at a constant speed $v$. Denoting by $r$ the radius of the paper roll at any given time and by $b$ the thickness of the paper, derive an expression for the angular acceleration of the paper roll.

15.37 Television recording tape is being rewound on a VCR reel which rotates with a constant angular velocity $\omega_0$. Denoting by $r$ the radius of the reel and tape at any given time and by $b$ the thickness of the tape, derive an expression for the acceleration of the tape as it approaches the reel.

15.5 GENERAL PLANE MOTION

As indicated in Sec. 15.1, we understand by general plane motion a plane motion which is neither a translation nor a rotation. As you will presently see, however, a general plane motion can always be considered as the sum of a translation and a rotation.

Consider, for example, a wheel rolling on a straight track (Fig. 15.12). Over a certain interval of time, two given points $A$ and $B$ will have moved, respectively, from $A_1$ to $A_2$ and from $B_1$ to $B_2$. The same result could be obtained through a translation which would bring $A$ and $B$ into $A_2$ and $B_1'$ (the line $AB$ remaining vertical), followed by a rotation about $A$ bringing $B$ into $B_2$. Although the original rolling motion differs from the combination of translation and rotation when these motions are taken in succession, the original motion can be exactly duplicated by a combination of simultaneous translation and rotation.
Another example of plane motion is given in Fig. 15.13, which represents a rod whose extremities slide along a horizontal and a vertical track, respectively. This motion can be replaced by a translation in a horizontal direction and a rotation about $A$ (Fig. 15.13a) or by a translation in a vertical direction and a rotation about $B$ (Fig. 15.13b).

In the general case of plane motion, we will consider a small displacement which brings two particles $A$ and $B$ of a representative slab, respectively, from $A_1$ and $B_1$ into $A_2$ and $B_2$ (Fig. 15.14). This displacement can be divided into two parts: in one, the particles move into $A_2$ and $B_1'$ while the line $AB$ maintains the same direction; in the other, $B$ moves into $B_2$ while $A$ remains fixed. The first part of the motion is clearly a translation and the second part a rotation about $A$.

Recalling from Sec. 11.12 the definition of the relative motion of a particle with respect to a moving frame of reference—as opposed to its absolute motion with respect to a fixed frame of reference—we can restate as follows the result obtained above: Given two particles $A$ and $B$ of a rigid slab in plane motion, the relative motion of $B$ with respect to a frame attached to $A$ and of fixed orientation is a rotation. To an observer moving with $A$ but not rotating, particle $B$ will appear to describe an arc of circle centered at $A$.
15.6 ABSOLUTE AND RELATIVE VELOCITY IN PLANE MOTION

We saw in the preceding section that any plane motion of a slab can be replaced by a translation defined by the motion of an arbitrary reference point \( A \) and a simultaneous rotation about \( A \). The absolute velocity \( \mathbf{v}_B \) of a particle \( B \) of the slab is obtained from the relative-velocity formula derived in Sec. 11.12,

\[
\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}
\]

where the right-hand member represents a vector sum. The velocity \( \mathbf{v}_A \) corresponds to the translation of the slab with \( A \), while the relative velocity \( \mathbf{v}_{B/A} \) is associated with the rotation of the slab about \( A \) and is measured with respect to axes centered at \( A \) and of fixed orientation (Fig. 15.15). Denoting by \( \mathbf{r}_{B/A} \) the position vector of \( B \) relative to \( A \), and by \( \mathbf{w} \) the angular velocity of the slab with respect to axes of fixed orientation, we have from (15.10) and (15.10')

\[
\mathbf{v}_{B/A} = \mathbf{w} \times \mathbf{r}_{B/A} \quad \mathbf{v}_{B/A} = r \mathbf{w}
\]

Fig. 15.15

where \( r \) is the distance from \( A \) to \( B \). Substituting for \( \mathbf{v}_{B/A} \) from (15.18) into (15.17), we can also write

\[
\mathbf{v}_B = \mathbf{v}_A + \mathbf{w} \times \mathbf{r}_{B/A} \quad (15.17')
\]

As an example, let us again consider the rod \( AB \) of Fig. 15.13. Assuming that the velocity \( \mathbf{v}_A \) of end \( A \) is known, we propose to find the velocity \( \mathbf{v}_B \) of end \( B \) and the angular velocity \( \mathbf{w} \) of the rod, in terms of the velocity \( \mathbf{v}_A \), the length \( l \), and the angle \( \theta \). Choosing \( A \) as a reference point, we express that the given motion is equivalent to a translation with \( A \) and a simultaneous rotation about \( A \) (Fig. 15.16). The absolute velocity of \( B \) must therefore be equal to the vector sum

\[
\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \quad (15.17)
\]

We note that while the direction of \( \mathbf{v}_{B/A} \) is known, its magnitude \( l \omega \) is unknown. However, this is compensated for by the fact that the direction of \( \mathbf{v}_B \) is known. We can therefore complete the diagram of Fig. 15.16. Solving for the magnitudes \( v_B \) and \( \omega \), we write

\[
v_B = v_A \tan \theta \quad \omega = \frac{v_{B/A}}{l} = \frac{v_A}{l \cos \theta}
\]

(15.19)
The same result can be obtained by using \( B \) as a point of reference. Resolving the given motion into a translation with \( B \) and a simultaneous rotation about \( B \) (Fig. 15.17), we write the equation

\[
v_A = v_B + v_{AB}
\]  

(15.20)

which is represented graphically in Fig. 15.17. We note that \( v_{AB} \) and \( v_{BA} \) have the same magnitude \( l\omega \) but opposite sense. The sense of the relative velocity depends, therefore, upon the point of reference which has been selected and should be carefully ascertained from the appropriate diagram (Fig. 15.16 or 15.17).

Finally, we observe that the angular velocity \( \omega \) of the rod in its rotation about \( B \) is the same as in its rotation about \( A \). It is measured in both cases by the rate of change of the angle \( \theta \). This result is quite general; we should therefore bear in mind that the angular velocity \( \omega \) of a rigid body in plane motion is independent of the reference point.

Most mechanisms consist not of one but of several moving parts. When the various parts of a mechanism are pin-connected, the analysis of the mechanism can be carried out by considering each part as a rigid body, keeping in mind that the points where two parts are connected must have the same absolute velocity (see Sample Prob. 15.3). A similar analysis can be used when gears are involved, since the teeth in contact must also have the same absolute velocity. However, when a mechanism contains parts which slide on each other, the relative velocity of the parts in contact must be taken into account (see Secs. 15.10 and 15.11).
SAMPLE PROBLEM 15.2

The double gear shown rolls on the stationary lower rack; the velocity of its center $A$ is 1.2 m/s directed to the right. Determine (a) the angular velocity of the gear, (b) the velocities of the upper rack $R$ and of point $D$ of the gear.

\[ v_A = 1.2 \text{ m/s} \]

\[ r_1 = 150 \text{ mm} \]

\[ r_2 = 100 \text{ mm} \]

\[
\begin{align*}
\text{SOLUTION} \\
\text{a. Angular Velocity of the Gear.} & \quad \text{Since the gear rolls on the lower rack, its center } A \text{ moves through a distance equal to the outer circumference } 2\pi r_1 \text{ for each full revolution of the gear. Noting that } 1 \text{ rev } = 2\pi \text{ rad, and that when } A \text{ moves to the right } (x_A > 0) \text{ the gear rotates clockwise } (\theta < 0), \text{ we write} \\
& \quad \frac{x_A}{2\pi r_1} = -\frac{\theta}{2\pi} \\
& \quad x_A = -r_1 \theta \\
\text{Differentiating with respect to the time } t \text{ and substituting the known values } v_A = 1.2 \text{ m/s and } r_1 = 150 \text{ mm } = 0.150 \text{ m, we obtain} \\
& \quad v_A = -r_1 \omega \quad 1.2 \text{ m/s } = -(0.150 \text{ m})\omega \quad \omega = -8 \text{ rad/s} \\
& \quad \omega = \omega \hat{k} = -(8 \text{ rad/s})\hat{k} \quad \text{ where } \hat{k} \text{ is a unit vector pointing out of the paper.} \\
\text{b. Velocities.} & \quad \text{The rolling motion is resolved into two component motions: a translation with the center } A \text{ and a rotation about the center } A. \text{ In the translation, all points of the gear move with the same velocity } v_A. \text{ In the rotation, each point } P \text{ of the gear moves about } A \text{ with a relative velocity } v_{P/A} = \omega \hat{k} \times r_{P/A}, \text{ where } r_{P/A} \text{ is the position vector of } P \text{ relative to } A. \\
\text{Velocity of Upper Rack.} & \quad \text{The velocity of the upper rack is equal to the velocity of point } B; \text{ we write} \\
& \quad v_R = v_B = v_A + v_{R/A} = v_A + \omega \hat{k} \times r_{R/A} \\
& \quad = (1.2 \text{ m/s})\hat{i} - (8 \text{ rad/s})\hat{k} \times (0.100 \text{ m})\hat{j} \\
& \quad = (1.2 \text{ m/s})\hat{i} + (0.8 \text{ m/s})\hat{i} = (2 \text{ m/s})\hat{i} \\
& \quad v_R = 2 \text{ m/s} \quad \boxed{} \\
\text{Velocity of Point } D & \quad v_D = v_A + v_{D/A} = v_A + \omega \hat{k} \times r_{D/A} \\
& \quad = (1.2 \text{ m/s})\hat{i} - (8 \text{ rad/s})\hat{k} \times (-0.150 \text{ m})\hat{i} \\
& \quad = (1.2 \text{ m/s})\hat{i} + (1.2 \text{ m/s})\hat{j} \\
& \quad v_D = 1.697 \text{ m/s } \angle 45^\circ \quad \boxed{}}
**SAMPLE PROBLEM 15.3**

In the engine system shown, the crank $AB$ has a constant clockwise angular velocity of 2000 rpm. For the crank position indicated, determine (a) the angular velocity of the connecting rod $BD$, (b) the velocity of the piston $P$.

---

**SOLUTION**

Motion of Crank $AB$. The crank $AB$ rotates about point $A$. Expressing $\omega_{AB}$ in rad/s and writing $v_B = r\omega_{AB}$, we obtain

$$\omega_{AB} = \left(\frac{2000 \, \text{rev}}{\text{min}}\right) \left(\frac{1 \, \text{min}}{60 \, \text{s}}\right) \left(\frac{2\pi \, \text{rad}}{1 \, \text{rev}}\right) = 209.4 \, \text{rad/s}$$

$$v_B = (AB)\omega_{AB} = (3 \, \text{in.})(209.4 \, \text{rad/s}) = 628.3 \, \text{in./s}$$

Motion of Connecting Rod $BD$. We consider this motion as a general plane motion. Using the law of sines, we compute the angle $\beta$ between the connecting rod and the horizontal:

$$\sin 40^\circ \frac{8 \, \text{in.}}{3 \, \text{in.}} = \sin \beta \Rightarrow \beta = 13.95^\circ$$

The velocity $v_D$ of the point $D$ where the rod is attached to the piston must be horizontal, while the velocity of point $B$ is equal to the velocity $v_B$ obtained above. Resolving the motion of $BD$ into a translation with $B$ and a rotation about $B$, we obtain

$$v_D = v_B + v_{DB}$$

Expressing the relation between the velocities $v_D$, $v_B$, and $v_{DB}$, we write

$$v_D = v_B + v_{DB}$$

We draw the vector diagram corresponding to this equation. Recalling that $\beta = 13.95^\circ$, we determine the angles of the triangle and write

$$\frac{v_D}{\sin 53.95^\circ} = \frac{v_{DB}}{\sin 50^\circ} = \frac{628.3 \, \text{in./s}}{\sin 76.05^\circ}$$

$$v_{DB} = 495.9 \, \text{in./s}$$

$$v_D = 523.4 \, \text{in./s} = 43.6 \, \text{ft/s}$$

Since $v_{DB} = l\omega_{BD}$, we have

$$495.9 \, \text{in./s} = (8 \, \text{in.})\omega_{BD} \Rightarrow \omega_{BD} = 62.0 \, \text{rad/s}$$
I
n this lesson you learned to analyze the velocity of bodies in general plane motion. You found that a general plane motion can always be considered as the sum of the two motions you studied in the last lesson, namely, a translation and a rotation.

To solve a problem involving the velocity of a body in plane motion you should take the following steps.

1. **Whenever possible determine the velocity of the points of the body** where the body is connected to another body whose motion is known. That other body may be an arm or crank rotating with a given angular velocity [Sample Prob. 15.3].

2. **Next start drawing a “diagram equation”** to use in your solution (Figs. 15.15 and 15.16). This “equation” will consist of the following diagrams.
   - **a. Plane motion diagram:** Draw a diagram of the body including all dimensions and showing those points for which you know or seek the velocity.
   - **b. Translation diagram:** Select a reference point A for which you know the direction and/or the magnitude of the velocity \( v_A \), and draw a second diagram showing the body in translation with all of its points having the same velocity \( v_A \).
   - **c. Rotation diagram:** Consider point A as a fixed point and draw a diagram showing the body in rotation about A. Show the angular velocity \( \omega = \omega \hat{k} \) of the body and the relative velocities with respect to A of the other points, such as the velocity \( v_{B/A} \) of B relative to A.

3. **Write the relative-velocity formula**
   \[
   v_B = v_A + v_{B/A}
   \]

   While you can solve this vector equation analytically by writing the corresponding scalar equations, you will usually find it easier to solve it by using a vector triangle (Fig. 15.16).

4. **A different reference point can be used to obtain an equivalent solution.** For example, if point B is selected as the reference point, the velocity of point A is expressed as
   \[
   v_A = v_B + v_{A/B}
   \]

   Note that the relative velocities \( v_{B/A} \) and \( v_{A/B} \) have the same magnitude but opposite sense. Relative velocities, therefore, depend upon the reference point that has been selected. The angular velocity, however, is independent of the choice of reference point.
15.38 The motion of rod AB is guided by pins attached at A and B which slide in the slots shown. At the instant shown, \( \theta = 40^\circ \) and the pin at B moves upward to the left with a constant velocity of 6 in./s. Determine (a) the angular velocity of the rod, (b) the velocity of the pin at end A.

15.39 The motion of rod AB is guided by pins attached at A and B which slide in the slots shown. At the instant shown, \( \theta = 30^\circ \) and the pin at A moves downward with a constant velocity of 9 in./s. Determine (a) the angular velocity of the rod, (b) the velocity of the pin at end B.

15.40 Small wheels have been attached to the ends of rod AB and roll freely along the surfaces shown. Knowing that wheel A moves to the left with a constant velocity of 1.5 m/s, determine (a) the angular velocity of the rod, (b) the velocity of end B of the rod.

15.41 Collar A moves upward with a constant velocity of 1.2 m/s. At the instant shown when \( \theta = 25^\circ \), determine (a) the angular velocity of rod AB, (b) the velocity of collar B.

15.42 Collar B moves downward to the left with a constant velocity of 1.6 m/s. At the instant shown when \( \theta = 40^\circ \), determine (a) the angular velocity of rod AB, (b) the velocity of collar A.
15.43 Rod AB moves over a small wheel at C while end A moves to the right with a constant velocity of 500 mm/s. At the instant shown, determine (a) the angular velocity of the rod, (b) the velocity of end B of the rod.

15.44 The plate shown moves in the xy plane. Knowing that \(v_A_x = 12 \text{ in./s}\), \(v_B_y = -4 \text{ in./s}\), and \(v_C_y = -24 \text{ in./s}\), determine (a) the angular velocity of the plate, (b) the velocity of point B.

15.45 In Prob. 15.44, determine (a) the velocity of point A, (b) the point on the plate with zero velocity.

15.46 The plate shown moves in the xy plane. Knowing that \(v_A_x = 120 \text{ mm/s}\), \(v_B_y = 300 \text{ mm/s}\), and \(v_C_y = -60 \text{ mm/s}\), determine (a) the angular velocity of the plate, (b) the velocity of point A.

15.47 In Prob. 15.46, determine (a) the velocity of point B, (b) the point of the plate with zero velocity.
15.48 In the planetary gear system shown, the radius of gears A, B, C, and D is 3 in. and the radius of the outer gear E is 9 in. Knowing that gear E has an angular velocity of 120 rpm clockwise and that the central gear has an angular velocity of 150 rpm clockwise, determine (a) the angular velocity of each planetary gear, (b) the angular velocity of the spider connecting the planetary gears.

Fig. P15.48 and P15.49

15.49 In the planetary gear system shown, the radius of the central gear A is \( a \), the radius of each of the planetary gears is \( b \), and the radius of the outer gear E is \( a + 2b \). The angular velocity of gear A is \( \omega_A \) clockwise, and the outer gear is stationary. If the angular velocity of the spider \( BCD \) is to be \( \omega_A / 5 \) clockwise, determine (a) the required value of the ratio \( b / a \), (b) the corresponding angular velocity of each planetary gear.

15.50 Gear A rotates with an angular velocity of 120 rpm clockwise. Knowing that the angular velocity of arm AB is 90 rpm clockwise, determine the corresponding angular velocity of gear B.

Fig. P15.50 and P15.51

15.51 Arm AB rotates with an angular velocity of 42 rpm clockwise. Determine the required angular velocity of gear A for which (a) the angular velocity of gear B is 20 rpm counterclockwise, (b) the motion of gear B is a curvilinear translation.

15.52 Arm AB rotates with an angular velocity of 20 rad/s counterclockwise. Knowing that the outer gear C is stationary, determine (a) the angular velocity of gear B, (b) the velocity of the gear tooth located at point D.

Fig. P15.52
15.53 and 15.54 Arm $ACB$ rotates about point $C$ with an angular velocity of $40$ rad/s counterclockwise. Two friction disks $A$ and $B$ are pinned at their centers to arm $ACB$ as shown. Knowing that the disks roll without slipping at surfaces of contact, determine the angular velocity of (a) disk $A$, (b) disk $B$.

15.55 Knowing that crank $AB$ has a constant angular velocity of $160$ rpm counterclockwise, determine the angular velocity of rod $BD$ and the velocity of collar $D$ when (a) $\theta = 0$, (b) $\theta = 90^\circ$.

15.56 Knowing that crank $AB$ has a constant angular velocity of $160$ rpm counterclockwise, determine the angular velocity of rod $BD$ and the velocity of collar $D$ when $\theta = 60^\circ$. 
15.57 In the engine system shown, \( l = 160 \text{ mm} \) and \( b = 60 \text{ mm} \). Knowing that the crank \( AB \) rotates with a constant angular velocity of 1000 rpm clockwise, determine the velocity of the piston \( P \) and the angular velocity of the connecting rod when (a) \( \theta = 0 \), (b) \( \theta = 90^\circ \).

15.58 In the engine system shown in Fig. P15.57 and P15.58, \( l = 160 \text{ mm} \) and \( b = 60 \text{ mm} \). Knowing that crank \( AB \) rotates with a constant angular velocity of 1000 rpm clockwise, determine the velocity of the piston \( F \) and the angular velocity of the connecting rod when \( \theta = 60^\circ \).

15.59 A straight rack rests on a gear of radius \( r \) and is attached to a block \( B \) as shown. Denoting by \( \omega_D \) the clockwise angular velocity of gear \( D \) and by \( \theta \) the angle formed by the rack and the horizontal, derive expressions for the velocity of block \( B \) and the angular velocity of the rack in terms of \( r \), \( \theta \), and \( \omega_D \).

15.60 A straight rack rests on a gear of radius \( r = 75 \text{ mm} \) and is attached to a block \( B \) as shown. Knowing that at the instant shown the angular velocity of gear \( D \) is 15 rpm counterclockwise and \( \theta = 20^\circ \), determine (a) the velocity of block \( B \), (b) the angular velocity of the rack.

15.61 A straight rack rests on a gear of radius \( r = 60 \text{ mm} \) and is attached to a block \( B \) as shown. Knowing that at the instant shown the velocity of block \( B \) is 200 mm/s to the right and \( \theta = 25^\circ \), determine (a) the angular velocity of gear \( D \), (b) the angular velocity of the rack.

15.62 In the eccentric shown, a disk of 2-in.-radius revolves about shaft \( O \) that is located 0.5 in. from the center \( A \) of the disk. The distance between the center \( A \) of the disk and the pin at \( B \) is 8 in. Knowing that the angular velocity of the disk is 900 rpm clockwise, determine the velocity of the block when \( \theta = 30^\circ \).
15.63 through 15.65 In the position shown, bar $AB$ has an angular velocity of 4 rad/s clockwise. Determine the angular velocity of bars $BD$ and $DE$.

15.66 In the position shown, bar $DE$ has a constant angular velocity of 10 rad/s clockwise. Knowing that $h = 500$ mm, determine (a) the angular velocity of bar $FBD$, (b) the velocity of point $F$. 

Fig. P15.63

Fig. P15.64

Fig. P15.65

Fig. P15.66 and P15.67
15.67 In the position shown, bar DE has a constant angular velocity of 10 rad/s clockwise. Determine (a) the distance $h$ for which the velocity of point $F$ is vertical, (b) the corresponding velocity of point $F$.

15.68 In the position shown, bar AB has zero angular acceleration and an angular velocity of 20 rad/s counterclockwise. Determine (a) the angular velocity of member BDH, (b) the velocity of point $G$.

![Diagram of bar AB and BDH with measurements: 3 in., 5 in., 5 in., 3 in., 4 in., 10 in., 5 in., 3 in.] Fig. P15.68 and P15.69

15.69 In the position shown, bar AB has zero angular acceleration and an angular velocity of 20 rad/s counterclockwise. Determine (a) the angular velocity of member BDH, (b) the velocity of point $H$.

15.70 An automobile travels to the right at a constant speed of 48 mi/h. If the diameter of a wheel is 22 in., determine the velocities of points B, C, D, and E on the rim of the wheel.

![Diagram of a wheel with points B, C, D, E, and measurements: 30°, 90°, 22 in., 90°] Fig. P15.70
15.71 The 80-mm-radius wheel shown rolls to the left with a velocity of 900 mm/s. Knowing that the distance AD is 50 mm, determine the velocity of the collar and the angular velocity of rod AB when (a) $\beta = 0$, (b) $\beta = 90^\circ$.

![Fig. P15.71](image)

15.72 For the gearing shown, derive an expression for the angular velocity $\omega_C$ of gear C and show that $\omega_C$ is independent of the radius of gear B. Assume that point A is fixed and denote the angular velocities of rod ABC and gear A by $\omega_{ABC}$ and $\omega_A$, respectively.

![Fig. P15.72](image)

15.7 INSTANTANEOUS CENTER OF ROTATION IN PLANE MOTION

Consider the general plane motion of a slab. We propose to show that at any given instant the velocities of the various particles of the slab are the same as if the slab were rotating about a certain axis perpendicular to the plane of the slab, called the *instantaneous axis of rotation*. This axis intersects the plane of the slab at a point C, called the *instantaneous center of rotation* of the slab.

We first recall that the plane motion of a slab can always be replaced by a translation defined by the motion of an arbitrary reference point A and by a rotation about A. As far as the velocities are concerned, the translation is characterized by the velocity $\mathbf{v}_A$ of the reference point A and the rotation is characterized by the angular velocity $\mathbf{\omega}$ of the slab (which is independent of the choice of A). Thus, the velocity $\mathbf{v}_A$ of point A and the angular velocity $\mathbf{\omega}$ of the slab define...
completely the velocities of all the other particles of the slab (Fig. 15.18a). Now let us assume that \( v_A \) and \( \omega \) are known and that they are both different from zero. (If \( v_A = 0 \), point \( A \) is itself the instantaneous center of rotation, and if \( \omega = 0 \), all the particles have the same velocity \( v_A \).) These velocities could be obtained by letting the slab rotate with the angular velocity \( \omega \) about a point \( C \) located on the perpendicular to \( v_A \) at a distance \( r = v_A/\omega \) from \( A \) as shown in Fig. 15.18b. We check that the velocity of \( A \) would be perpendicular to \( AC \) and that its magnitude would be \( r\omega = (v_A/\omega)\omega = v_A \). Thus the velocities of all the other particles of the slab would be the same as originally defined. Therefore, as far as the velocities are concerned, the slab seems to rotate about the instantaneous center \( C \) at the instant considered.

The position of the instantaneous center can be defined in two other ways. If the directions of the velocities of two particles \( A \) and \( B \) of the slab are known and if they are different, the instantaneous center \( C \) is obtained by drawing the perpendicular to \( v_A \) through \( A \) and the perpendicular to \( v_B \) through \( B \) and determining the point in which these two lines intersect (Fig. 15.19a). If the velocities \( v_A \) and \( v_B \) of two particles \( A \) and \( B \) are perpendicular to the line \( AB \) and if their magnitudes are known, the instantaneous center can be found by intersecting the line \( AB \) with the line joining the extremities of the vectors \( v_A \) and \( v_B \) (Fig. 15.19b). Note that if \( v_A \) and \( v_B \) were parallel
in Fig. 15.19a or if \( \mathbf{v}_A \) and \( \mathbf{v}_B \) had the same magnitude in Fig. 15.19b, the instantaneous center \( C \) would be at an infinite distance and \( \omega \) would be zero; all points of the slab would have the same velocity.

To see how the concept of instantaneous center of rotation can be put to use, let us consider again the rod of Sec. 15.6. Drawing the perpendicular to \( \mathbf{v}_A \) through \( A \) and the perpendicular to \( \mathbf{v}_B \) through \( B \) (Fig. 15.20), we obtain the instantaneous center \( C \). At the instant considered, the velocities of all the particles of the rod are thus the same as if the rod rotated about \( C \). Now, if the magnitude \( v_A \) of the velocity of \( A \) is known, the magnitude \( \omega \) of the angular velocity of the rod can be obtained by writing

\[
\omega = \frac{v_A}{AC} = \frac{v_A}{l \cos \theta}
\]

The magnitude of the velocity of \( B \) can then be obtained by writing

\[
v_B = (BC)\omega = l \sin \theta \frac{v_A}{l \cos \theta} = v_A \tan \theta
\]

Note that only absolute velocities are involved in the computation.

The instantaneous center of a slab in plane motion can be located either on the slab or outside the slab. If it is located on the slab, the particle \( C \) coinciding with the instantaneous center at a given instant \( t \) must have zero velocity at that instant. However, it should be noted that the instantaneous center of rotation is valid only at a given instant. Thus, the particle \( C \) of the slab which coincides with the instantaneous center at time \( t \) will generally not coincide with the instantaneous center at time \( t + \Delta t \); while its velocity is zero at time \( t \), it will probably be different from zero at time \( t + \Delta t \). This means that, in general, the particle \( C \) does not have zero acceleration and, therefore, that the accelerations of the various particles of the slab cannot be determined as if the slab were rotating about \( C \).

As the motion of the slab proceeds, the instantaneous center moves in space. But it was just pointed out that the position of the instantaneous center on the slab keeps changing. Thus, the instantaneous center describes one curve in space, called the space centrode, and another curve on the slab, called the body centrode (Fig. 15.21). It can be shown that at any instant, these two curves are tangent at \( C \) and that as the slab moves, the body centrode appears to roll on the space centrode.
SAMPLE PROBLEM 15.4

Solve Sample Prob. 15.2, using the method of the instantaneous center of rotation.

SOLUTION

a. Angular Velocity of the Gear. Since the gear rolls on the stationary lower rack, the point of contact C of the gear with the rack has no velocity; point C is therefore the instantaneous center of rotation. We write

\[ v_A = r_A \omega = 1.2 \text{ m/s} = (0.150 \text{ m}) \omega \]

\[ \omega = 8 \text{ rad/s} \triangleleft \]

b. Velocities. As far as velocities are concerned, all points of the gear seem to rotate about the instantaneous center.

Velocity of Upper Rack. Recalling that \( v_R = v_B \), we write

\[ v_R = v_B = r_B \omega = (0.250 \text{ m})(8 \text{ rad/s}) = 2 \text{ m/s} \rightarrow \triangleleft \]

Velocity of Point D. Since \( r_D = (0.150 \text{ m})\sqrt{2} = 0.2121 \text{ m} \), we write

\[ v_D = r_D \omega = (0.2121 \text{ m})(8 \text{ rad/s}) = 1.697 \text{ m/s} \]

\[ v_D = 1.697 \text{ m/s} \triangleleft 45^\circ \]

SAMPLE PROBLEM 15.5

Solve Sample Prob. 15.3, using the method of the instantaneous center of rotation.

SOLUTION

Motion of Crank AB. Referring to Sample Prob. 15.3, we obtain the velocity of point B; \( v_B = 628.3 \text{ in./s} \triangleleft 50^\circ \).

Motion of the Connecting Rod BD. We first locate the instantaneous center C by drawing lines perpendicular to the absolute velocities \( v_B \) and \( v_D \). Recalling from Sample Prob. 15.3 that \( \beta = 13.95^\circ \) and that \( BD = 8 \text{ in.} \), we solve the triangle BCD.

\[ \gamma_B = 40^\circ + \beta = 53.95^\circ \]

\[ \gamma_D = 90^\circ - \beta = 76.05^\circ \]

\[ \frac{BC}{\sin 76.05^\circ} = \frac{CD}{\sin 53.95^\circ} = \frac{8 \text{ in.}}{\sin 50^\circ} \]

\[ BC = 10.14 \text{ in.} \quad CD = 8.44 \text{ in.} \]

Since the connecting rod BD seems to rotate about point C, we write

\[ v_B = (BC) \omega_{BD} \]

\[ 628.3 \text{ in./s} = (10.14 \text{ in.}) \omega_{BD} \]

\[ \omega_{BD} = 62 \text{ rad/s} \triangleleft \]

\[ v_D = (CD) \omega_{BD} = (8.44 \text{ in.})(62 \text{ rad/s}) \]

\[ = 523 \text{ in./s} = 43.6 \text{ ft/s} \rightarrow \]

\[ v_D = v_D = 43.6 \text{ ft/s} \rightarrow \triangleleft \]
In this lesson we introduced the instantaneous center of rotation in plane motion. This provides us with an alternative way for solving problems involving the velocities of the various points of a body in plane motion.

As its name suggests, the instantaneous center of rotation is the point about which you can assume a body is rotating at a given instant, as you determine the velocities of the points of the body at that instant.

A. To determine the instantaneous center of rotation of a body in plane motion, you should use one of the following procedures.

1. If the velocity $v_A$ of a point $A$ and the angular velocity $\omega$ of the body are both known (Fig. 15.18):
   a. Draw a sketch of the body, showing point $A$, its velocity $v_A$, and the angular velocity $\omega$ of the body.
   b. From $A$ draw a line perpendicular to $v_A$ on the side of $v_A$ from which this velocity is viewed as having the same sense as $\omega$.
   c. Locate the instantaneous center $C$ on this line, at a distance $r = v_A/\omega$ from point $A$.

2. If the directions of the velocities of two points $A$ and $B$ are known and are different (Fig. 15.19a):
   a. Draw a sketch of the body, showing points $A$ and $B$ and their velocities $v_A$ and $v_B$.
   b. From $A$ and $B$ draw lines perpendicular to $v_A$ and $v_B$, respectively. The instantaneous center $C$ is located at the point where the two lines intersect.
   c. If the velocity of one of the two points is known, you can determine the angular velocity of the body. For example, if you know $v_A$, you can write $\omega = v_A/AC$, where $AC$ is the distance from point $A$ to the instantaneous center $C$.

3. If the velocities of two points $A$ and $B$ are known and are both perpendicular to the line $AB$ (Fig. 15.19b):
   a. Draw a sketch of the body, showing points $A$ and $B$ with their velocities $v_A$ and $v_B$ drawn to scale.
   b. Draw a line through points $A$ and $B$, and another line through the tips of the vectors $v_A$ and $v_B$. The instantaneous center $C$ is located at the point where the two lines intersect.
c. The angular velocity of the body is obtained by either dividing $v_A$ by $AC$ or $v_B$ by $BC$.

d. If the velocities $v_A$ and $v_B$ have the same magnitude, the two lines drawn in part b do not intersect; the instantaneous center $C$ is at an infinite distance. The angular velocity $\omega$ is zero and the body is in translation.

B. Once you have determined the instantaneous center and the angular velocity of a body, you can determine the velocity $v_P$ of any point $P$ of the body in the following way.

1. Draw a sketch of the body, showing point $P$, the instantaneous center of rotation $C$, and the angular velocity $\omega$.

2. Draw a line from $P$ to the instantaneous center $C$ and measure or calculate the distance from $P$ to $C$.

3. The velocity $v_P$ is a vector perpendicular to the line $PC$, of the same sense as $\omega$, and of magnitude $v_P = (PC)\omega$.

Finally, keep in mind that the instantaneous center of rotation can be used only to determine velocities. It cannot be used to determine accelerations.
15.73 A 10-ft beam $AE$ is being lowered by means of two overhead cranes. At the instant shown it is known that the velocity of point $D$ is 24 in./s downward and the velocity of point $E$ is 36 in./s downward. Determine (a) the instantaneous center of rotation of the beam, (b) the velocity of point $A$.

15.74 A helicopter moves horizontally in the $x$ direction at a speed of 120 mi/h. Knowing that the main blades rotate clockwise with an angular velocity of 180 rpm, determine the instantaneous axis of rotation of the main blades.

15.75 The spool of tape shown and its frame assembly are pulled upward at a speed $v_A = 750$ mm/s. Knowing that the 80-mm-radius spool has an angular velocity of 15 rad/s clockwise and that at the instant shown the total thickness of the tape on the spool is 20 mm, determine (a) the instantaneous center of rotation of the spool, (b) the velocities of points $B$ and $D$.

15.76 The spool of tape shown and its frame assembly are pulled upward at a speed $v_A = 100$ mm/s. Knowing that end $B$ of the tape is pulled downward with a velocity of 300 mm/s and that at the instant shown the total thickness of the tape on the spool is 20 mm, determine (a) the instantaneous center of rotation of the spool, (b) the velocity of point $D$ of the spool.
15.77 Solve Sample Prob. 15.2, assuming that the lower rack is not stationary but moves to the left with a velocity of 0.6 m/s.

15.78 A double pulley is attached to a slider block by a pin at A. The 30-mm-radius inner pulley is rigidly attached to the 60-mm-radius outer pulley. Knowing that each of the two cords is pulled at a constant speed as shown, determine (a) the instantaneous center of rotation of the double pulley, (b) the velocity of the slider block, (c) the number of millimeters of cord wrapped or unwrapped on each pulley per second.

15.79 Solve Prob. 15.78, assuming that cord E is pulled upward at a speed of 160 mm/s and cord F is pulled downward at a speed of 200 mm/s.

15.80 and 15.81 A 3-in.-radius drum is rigidly attached to a 5-in.-radius drum as shown. One of the drums rolls without sliding on the surface shown, and a cord is wound around the other drum. Knowing that end E of the cord is pulled to the left with a velocity of 6 in./s, determine (a) the angular velocity of the drums, (b) the velocity of the center of the drums, (c) the length of cord wound or unwound per second.

15.82 Knowing that at the instant shown the angular velocity of rod AB is 15 rad/s clockwise, determine (a) the angular velocity of rod BD, (b) the velocity of the midpoint of rod BD.

15.83 Knowing that at the instant shown the velocity of point D is 2.4 m/s upward, determine (a) the angular velocity of rod AB, (b) the velocity of the midpoint of rod BD.
15.84 Rod $ABD$ is guided by wheels at $A$ and $B$ that roll in horizontal and vertical tracks. Knowing that at the instant $\beta = 60^\circ$ and the velocity of wheel $B$ is 40 in./s downward, determine (a) the angular velocity of the rod, (b) the velocity of point $D$.

15.85 An overhead door is guided by wheels at $A$ and $B$ that roll in horizontal and vertical tracks. Knowing that when $\theta = 40^\circ$ the velocity of wheel $B$ is 1.5 ft/s upward, determine (a) the angular velocity of the door, (b) the velocity of end $D$ of the door.

15.86 Knowing that at the instant shown the angular velocity of rod $BE$ is 4 rad/s counterclockwise, determine (a) the angular velocity of rod $AD$, (b) the velocity of collar $D$, (c) the velocity of point $A$.

15.87 Knowing that at the instant shown the velocity of collar $D$ is 1.6 m/s upward, determine (a) the angular velocity of rod $AD$, (b) the velocity of point $B$, (c) the velocity of point $A$.

15.88 Rod $AB$ can slide freely along the floor and the inclined plane. Denoting by $v_A$ the velocity of point $A$, derive an expression for (a) the angular velocity of the rod, (b) the velocity of end $B$.

15.89 Rod $AB$ can slide freely along the floor and the inclined plane. Knowing that $\theta = 20^\circ$, $\beta = 50^\circ$, $l = 0.6$ m, and $v_A = 3$ m/s, determine (a) the angular velocity of the rod, (b) the velocity of end $B$. 

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Fig. P15.84

Fig. P15.85

Fig. P15.86 and P15.87

Fig. P15.88 and P15.89

Fig. P15.89
15.90 Arm ABD is connected by pins to a collar at B and to crank DE. Knowing that the velocity of collar B is 400 mm/s upward, determine (a) the angular velocity of arm ABD, (b) the velocity of point A.

![Fig. P15.90 and P15.91](image)

15.91 Arm ABD is connected by pins to a collar at B and to crank DE. Knowing that the angular velocity of crank DE is 1.2 rad/s counterclockwise, determine (a) the angular velocity of arm ABD, (b) the velocity of point A.

15.92 Two slots have been cut in plate FG and the plate has been placed so that the slots fit two fixed pins A and B. Knowing that at the instant shown the angular velocity of crank DE is 6 rad/s clockwise, determine (a) the velocity of point F, (b) the velocity of point G.

15.93 Two identical rods ABF and DBE are connected by a pin at B. Knowing that at the instant shown the velocity of point D is 10 in./s upward, determine the velocity of (a) point E, (b) point F.

15.94 Rod AB is attached to a collar at A and is fitted with a small wheel at B. Knowing that when \( \theta = 60^\circ \) the velocity of the collar is 250 mm/s upward, determine (a) the angular velocity of rod AB, (b) the velocity of point B.

![Fig. P15.94](image)
15.95 Two collars $C$ and $D$ move along the vertical rod shown. Knowing that the velocity of the collar $C$ is $660$ mm/s downward, determine (a) the velocity of collar $D$, (b) the angular velocity of member $AB$.

15.96 Two $500$-mm rods are pin-connected at $D$ as shown. Knowing that $B$ moves to the left with a constant velocity of $360$ mm/s, determine at the instant shown (a) the angular velocity of each rod, (b) the velocity of $E$.

15.97 Two rods $AB$ and $DE$ are connected as shown. Knowing that point $D$ moves to the left with a velocity of $40$ in./s, determine (a) the angular velocity of each rod, (b) the velocity of point $A$.

15.98 Two rods $AB$ and $DE$ are connected as shown. Knowing that point $B$ moves downward with a velocity of $60$ in./s, determine (a) the angular velocity of each rod, (b) the velocity of point $E$.

15.99 Describe the space centrode and the body centrode of rod $ABD$ of Prob. 15.84. (Hint: The body centrode need not lie on a physical portion of the rod.)

15.100 Describe the space centrode and the body centrode of the gear of Sample Prob. 15.2 as the gear rolls on the stationary horizontal rack.

15.101 Using the method of Sec. 15.7, solve Prob. 15.62.

15.102 Using the method of Sec. 15.7, solve Prob. 15.64.

15.103 Using the method of Sec. 15.7, solve Prob. 15.65.

15.104 Using the method of Sec. 15.7, solve Prob. 15.70.
15.8 Absolute and Relative Acceleration in Plane Motion

We saw in Sec. 15.5 that any plane motion can be replaced by a translation defined by the motion of an arbitrary reference point \( A \) and a simultaneous rotation about \( A \). This property was used in Sec. 15.6 to determine the velocity of the various points of a moving slab. The same property will now be used to determine the acceleration of the points of the slab.

We first recall that the absolute acceleration \( \mathbf{a}_B \) of a particle of the slab can be obtained from the relative-acceleration formula derived in Sec. 11.12,

\[
\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \tag{15.21}
\]

where the right-hand member represents a vector sum. The acceleration \( \mathbf{a}_A \) corresponds to the translation of the slab with \( A \), while the relative acceleration \( \mathbf{a}_{B/A} \) is associated with the rotation of the slab about \( A \) and is measured with respect to axes centered at \( A \) and of fixed orientation. We recall from Sec. 15.3 that the relative acceleration \( \mathbf{a}_{B/A} \) can be resolved into two components, a \textit{tangential component} \( \mathbf{a}_{B/A}^t \) perpendicular to the line \( AB \), and a \textit{normal component} \( \mathbf{a}_{B/A}^n \) directed toward \( A \) (Fig. 15.22). Denoting by \( \mathbf{r}_{B/A} \) the position vector of \( B \) relative to \( A \) and, respectively, by \( \omega \mathbf{k} \) and \( \alpha \mathbf{k} \) the angular velocity and angular acceleration of the slab with respect to axes of fixed orientation, we have

\[
\begin{align*}
\mathbf{a}_{B/A}^t &= \alpha \mathbf{k} \times \mathbf{r}_{B/A} \\
\mathbf{a}_{B/A}^n &= -\omega^2 \mathbf{r}_{B/A}
\end{align*}
\]

where \( r \) is the distance from \( A \) to \( B \). Substituting into (15.21) the expressions obtained for the tangential and normal components of \( \mathbf{a}_{B/A} \), we can also write

\[
\mathbf{a}_B = \mathbf{a}_A + \alpha \mathbf{k} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A} \tag{15.21'}
\]

**Fig. 15.22**

Photo 15.6 The central gear rotates about a fixed axis and is pin-connected to three bars which are in general plane motion.
As an example, let us again consider the rod \( AB \) whose extremities slide, respectively, along a horizontal and a vertical track (Fig. 15.23). Assuming that the velocity \( \mathbf{v}_A \) and the acceleration \( \mathbf{a}_A \) of \( A \) are known, we propose to determine the acceleration \( \mathbf{a}_B \) of \( B \) and the angular acceleration \( \alpha \) of the rod. Choosing \( A \) as a reference point, we express that the given motion is equivalent to a translation with \( A \) and a rotation about \( A \). The absolute acceleration of \( B \) must be equal to the sum

\[
\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}
\]

where \( \mathbf{a}_{B/A} \) has the magnitude \( l \omega^2 \) and is directed toward \( A \), while \( \mathbf{a}_{B/A} \) has the magnitude \( l \alpha \) and is perpendicular to \( AB \). Students should note that there is no way to tell whether the tangential component \( \mathbf{a}_{B/A} \) is directed to the left or to the right, and therefore both possible directions for this component are indicated in Fig. 15.23. Similarly, both possible senses for \( \mathbf{a}_B \) are indicated, since it is not known whether point \( B \) is accelerated upward or downward.

Equation (15.23) has been expressed geometrically in Fig. 15.24. Four different vector polygons can be obtained, depending upon the sense of \( \mathbf{a}_A \) and the relative magnitude of \( \mathbf{a}_A \) and \( \mathbf{a}_{B/A} \). If we are to determine \( \mathbf{a}_B \) and \( \alpha \) from one of these diagrams, we must know not only \( \mathbf{a}_A \) and \( \theta \) but also \( a \). The angular velocity of the rod should therefore be separately determined by one of the methods indicated in Secs. 15.6 and 15.7. The values of \( \mathbf{a}_B \) and \( \alpha \) can then be obtained by considering successively the \( x \) and \( y \) components of the vectors shown in Fig. 15.24. In the case of polygon \( a \), for example, we write

\[
\begin{align*}
\text{\( x \) components:} & \quad 0 = a_A + l \omega^2 \sin \theta - l \alpha \cos \theta \\
\text{\( y \) components:} & \quad -a_B = -l \omega^2 \cos \theta - l \alpha \sin \theta
\end{align*}
\]

and solve for \( a_B \) and \( \alpha \). The two unknowns can also be obtained by direct measurement on the vector polygon. In that case, care should be taken to draw first the known vectors \( \mathbf{a}_A \) and \( \mathbf{a}_{B/A} \).

It is quite evident that the determination of accelerations is considerably more involved than the determination of velocities. Yet...
in the example considered here, the extremities A and B of the rod were moving along straight tracks, and the diagrams drawn were relatively simple. If A and B had moved along curved tracks, it would have been necessary to resolve the accelerations \( a_A \) and \( a_B \) into normal and tangential components and the solution of the problem would have involved six different vectors.

When a mechanism consists of several moving parts which are pin-connected, the analysis of the mechanism can be carried out by considering each part as a rigid body, keeping in mind that the points at which two parts are connected must have the same absolute acceleration (see Sample Prob. 15.7). In the case of meshed gears, the tangential components of the accelerations of the teeth in contact are equal, but their normal components are different.

**15.9 Analysis of Plane Motion in Terms of a Parameter**

In the case of certain mechanisms, it is possible to express the coordinates \( x \) and \( y \) of all the significant points of the mechanism by means of simple analytic expressions containing a single parameter. It is sometimes advantageous in such a case to determine the absolute velocity and the absolute acceleration of the various points of the mechanism directly, since the components of the velocity and of the acceleration of a given point can be obtained by differentiating the coordinates \( x \) and \( y \) of that point.

Let us consider again the rod \( AB \) whose extremities slide, respectively, in a horizontal and a vertical track (Fig. 15.25). The coordinates \( x_A \) and \( y_B \) of the extremities of the rod can be expressed in terms of the angle \( \theta \) the rod forms with the vertical:

\[
\begin{align*}
  x_A &= l \sin \theta & y_B &= l \cos \theta
\end{align*}
\]

(15.24)

Differentiating Eqs. (15.24) twice with respect to \( t \), we write

\[
\begin{align*}
  v_A &= \dot{x}_A = l \dot{\theta} \cos \theta & v_B &= \dot{y}_B = l \dot{\theta} \sin \theta \\
  a_A &= \ddot{x}_A = -l \ddot{\theta} \sin \theta + l \dot{\theta} \cos \theta & a_B &= \ddot{y}_B = -l \ddot{\theta} \cos \theta - l \dot{\theta} \sin \theta
\end{align*}
\]

Recalling that \( \dot{\theta} = \omega \) and \( \ddot{\theta} = \alpha \), we obtain

\[
\begin{align*}
  v_A &= l \omega \cos \theta & v_B &= -l \omega \sin \theta \\
  a_A &= -l \omega^2 \sin \theta + l \alpha \cos \theta & a_B &= -l \omega^2 \cos \theta - l \alpha \sin \theta
\end{align*}
\]

(15.25)

(15.26)

We note that a positive sign for \( v_A \) or \( a_A \) indicates that the velocity \( v_A \) or the acceleration \( a_A \) is directed to the right; a positive sign for \( v_B \) or \( a_B \) indicates that \( v_B \) or \( a_B \) is directed upward. Equations (15.25) can be used, for example to determine \( v_B \) and \( \omega \) when \( v_A \) and \( \theta \) are known. Substituting for \( \omega \) in (15.26), we can then determine \( a_B \) and \( \alpha \) if \( a_A \) is known.
SAMPLE PROBLEM 15.6

The center of the double gear of Sample Prob. 15.2 has a velocity of 1.2 m/s to the right and an acceleration of 3 m/s² to the right. Recalling that the lower rack is stationary, determine (a) the angular acceleration of the gear, (b) the acceleration of points B, C, and D of the gear.

SOLUTION

a. Angular Acceleration of the Gear. In Sample Prob. 15.2, we found that \( v_A = -r_1 \omega \) and \( v_A = -r_1 \alpha \). Differentiating the latter with respect to time, we obtain \( a_A = -r_1 \alpha \).

\[
\begin{align*}
v_A &= -r_1 \omega \\
a_A &= -r_1 \alpha
\end{align*}
\]

\[
\begin{align*}
v_A &= 1.2 \text{ m/s} \\
a_A &= 3 \text{ m/s}^2
\end{align*}
\]

\[
\begin{align*}
\omega &= -8 \text{ rad/s} \\
\alpha &= 20 \text{ rad/s}^2
\end{align*}
\]

b. Accelerations. The rolling motion of the gear is resolved into a translation with \( A \) and a rotation about \( A \).

\[
\text{Translation} + \text{Rotation} = \text{Rolling motion}
\]

Acceleration of Point B. Adding vectorially the accelerations corresponding to the translation and to the rotation, we obtain

\[
\begin{align*}
a_B &= a_A + \omega \times r_{BA} = a_A + \omega \times (r_{BA})_f + (r_{BA})_n \\
&= a_A + \omega \times r_{BA} - \omega^2 r_{BA} \\
&= (3 \text{ m/s}^2) \hat{i} - (20 \text{ rad/s}^2) \hat{k} \times (0.100 \text{ m}) \hat{j} - (8 \text{ rad/s}^2)(0.100 \text{ m}) \hat{j} \\
&= (3 \text{ m/s}^2) \hat{i} + (2 \text{ m/s}^2) \hat{i} - (6.40 \text{ m/s}^2) \hat{j}
\end{align*}
\]

\[
a_B = 8.12 \text{ m/s}^2 \angle 52.0°
\]

Acceleration of Point C

\[
\begin{align*}
a_C &= a_A + \omega \times \omega \times r_{CA} - \omega^2 r_{CA} \\
&= (3 \text{ m/s}^2) \hat{i} - (20 \text{ rad/s}^2) \hat{k} \times (-0.150 \text{ m}) \hat{j} - (8 \text{ rad/s}^2)(-0.150 \text{ m}) \hat{j} \\
&= (3 \text{ m/s}^2) \hat{i} - (3 \text{ m/s}^2) \hat{i} + (9.60 \text{ m/s}^2) \hat{j}
\end{align*}
\]

\[
a_C = 9.60 \text{ m/s}^2 \uparrow
\]

Acceleration of Point D

\[
\begin{align*}
a_D &= a_A + \omega \times \omega \times r_{DA} - \omega^2 r_{DA} \\
&= (3 \text{ m/s}^2) \hat{i} - (20 \text{ rad/s}^2) \hat{k} \times (-0.150 \text{ m}) \hat{j} - (8 \text{ rad/s}^2)(-0.150 \text{ m}) \hat{i} \\
&= (3 \text{ m/s}^2) \hat{i} + (3 \text{ m/s}^2) \hat{i} + (9.60 \text{ m/s}^2) \hat{i}
\end{align*}
\]

\[
a_D = 12.95 \text{ m/s}^2 \uparrow 13.4°
\]
SAMPLE PROBLEM 15.7

Crank AB of the engine system of Sample Prob. 15.3 has a constant clockwise angular velocity of 2000 rpm. For the crank position shown, determine the angular acceleration of the connecting rod BD and the acceleration of point D.

**SOLUTION**

**Motion of Crank AB.** Since the crank rotates about A with constant \( \omega_{AB} = 2000 \text{ rpm} = 209.4 \text{ rad/s} \), we have \( a_A = 0 \). The acceleration of \( B \) is therefore directed toward \( A \) and has a magnitude
\[
a_B = r \omega_A^2 = (\frac{3}{12} \text{ ft})(209.4 \text{ rad/s})^2 = 10,962 \text{ ft/s}^2
\]
\[
a_B = 10,962 \text{ ft/s}^2 \downarrow 40^\circ
\]

**Motion of the Connecting Rod BD.** The angular velocity \( \omega_{BD} \) and the value of \( \beta \) were obtained in Sample Prob. 15.3:
\[
\omega_{BD} = 62.0 \text{ rad/s} \uparrow \beta = 13.95^\circ
\]
The motion of BD is resolved into a translation with \( B \) and a rotation about \( B \). The relative acceleration \( a_{DB} \) is resolved into normal and tangential components:
\[
(a_{DB})_n = (BD)\omega_{BD}^2 = (\frac{8}{12} \text{ ft})(62.0 \text{ rad/s})^2 = 2563 \text{ ft/s}^2
\]
\[
(a_{DB})_t = (BD)\alpha_{BD} = (\frac{8}{12} \text{ ft})(0.6667 \text{ rad/s}^2) \downarrow 13.95^\circ
\]
\[
(a_{DB})_h = 0.6667 \alpha_{BD} \downarrow 76.05^\circ
\]

While \( (a_{DB})_h \) must be perpendicular to BD, its sense is not known.

Plane motion

\[
\begin{align*}
\text{Translation} & \quad + \quad \text{Rotation} \\
\end{align*}
\]

Noting that the acceleration \( a_D \) must be horizontal, we write
\[
a_D = a_B + a_{DB} = a_B + (a_{DB})_n + (a_{DB})_h
\]
\[
[a_D] = [10,962 \downarrow 40^\circ] + [2563 \downarrow 13.95^\circ] + [0.6667 \alpha_{BD} \downarrow 76.05^\circ]
\]

Equating \( x \) and \( y \) components, we obtain the following scalar equations:

\[
\begin{align*}
-a_D &= -10,962 \cos 40^\circ - 2563 \cos 13.95^\circ + 0.6667 \alpha_{BD} \sin 13.95^\circ \\
0 &= -10,962 \sin 40^\circ + 2563 \sin 13.95^\circ + 0.6667 \alpha_{BD} \cos 13.95^\circ
\end{align*}
\]

Solving the equations simultaneously, we obtain \( \alpha_{BD} = 9940 \text{ rad/s}^2 \) and \( a_D = 9290 \text{ ft/s}^2 \). The positive signs indicate that the senses shown on the vector polygon are correct; we write
\[
\begin{align*}
\alpha_{BD} &= 9940 \text{ rad/s}^2 \uparrow \\
a_D &= 9290 \text{ ft/s}^2 \leftarrow
\end{align*}
\]
SAMPLE PROBLEM 15.8

The linkage ABDE moves in the vertical plane. Knowing that in the position shown crank AB has a constant angular velocity \( \omega_1 \) of 20 rad/s counterclockwise, determine the angular velocities and angular accelerations of the connecting rod BD and of the crank DE.

SOLUTION

This problem could be solved by the method used in Sample Prob. 15.7. In this case, however, the vector approach will be used. The position vectors \( r_B, r_D, \) and \( r_{DB} \) are chosen as shown in the sketch.

**Velocities.** Since the motion of each element of the linkage is contained in the plane of the figure, we have

\[
\begin{align*}
\omega_{AB} &= \omega_{AB} \mathbf{r}_B = (20 \text{ rad/s}) \mathbf{k} & \omega_{BD} &= \omega_{BD} \mathbf{r}_D = \omega_{BD} \mathbf{k} & \omega_{DE} &= \omega_{DE} \mathbf{r}_D = \omega_{DE} \mathbf{k}
\end{align*}
\]

where \( \mathbf{k} \) is a unit vector pointing out of the paper. We now write

\[
\begin{align*}
\mathbf{v}_B &= \mathbf{v}_B + \omega_{BD} \times \mathbf{r}_D & \omega_{BD} \times \mathbf{r}_D &= \omega_{AB} \mathbf{r}_B + \omega_{BD} \mathbf{r}_D \\
\omega_{DE} \mathbf{r}_D &= (17 \mathbf{i} + 17 \mathbf{j}) = 20 \mathbf{k} \times (5 \mathbf{i} + 14 \mathbf{j}) + \omega_{BD} \mathbf{r}_D \times (12 \mathbf{i} + 3 \mathbf{j}) \\
&-17 \omega_{DE} \mathbf{j} - 17 \omega_{DE} \mathbf{i} = 160 \mathbf{j} - 280 \mathbf{i} + 12 \omega_{BD} \mathbf{j} - 3 \omega_{BD} \mathbf{i}
\end{align*}
\]

Equating the coefficients of the unit vectors \( \mathbf{i} \) and \( \mathbf{j} \), we obtain the following two scalar equations:

\[
\begin{align*}
-17 \omega_{DE} &= -280 - 3 \omega_{BD} \\
-17 \omega_{DE} &= +160 + 12 \omega_{BD} \\
\omega_{BD} &= -(29.33 \text{ rad/s}) \mathbf{k} & \omega_{DE} &= (11.29 \text{ rad/s}) \mathbf{k}
\end{align*}
\]

**Accelerations.** Noting that at the instant considered crank AB has a constant angular velocity, we write

\[
\begin{align*}
\alpha_{AB} &= 0 & \alpha_{BD} &= \alpha_{BD} \mathbf{r}_B = \alpha_{BD} \mathbf{k} & \alpha_{DE} &= \alpha_{DE} \mathbf{r}_D = \alpha_{DE} \mathbf{k} \\
\mathbf{a}_D &= \mathbf{a}_B + \omega_{BD} \mathbf{r}_B + \alpha_{BD} \mathbf{r}_B + \alpha_{DE} \mathbf{r}_D = \alpha_{BD} \mathbf{r}_B + \alpha_{DE} \mathbf{r}_D
\end{align*}
\]

Each term of Eq. (1) is evaluated separately:

\[
\begin{align*}
\mathbf{a}_D &= \alpha_{BD} \mathbf{r}_B = -17 \omega_{DE} \mathbf{i} - 17 \omega_{DE} \mathbf{j} + 5600 \mathbf{j} \\
\mathbf{a}_D &= \alpha_{BD} \mathbf{r}_B = -17 \omega_{DE} \mathbf{i} - 17 \omega_{DE} \mathbf{j} + 5600 \mathbf{j} \\
\mathbf{a}_D &= \alpha_{BD} \mathbf{r}_B = -17 \omega_{DE} \mathbf{i} - 17 \omega_{DE} \mathbf{j} + 5600 \mathbf{j} \\
\mathbf{a}_D &= \alpha_{BD} \mathbf{r}_B = -17 \omega_{DE} \mathbf{i} - 17 \omega_{DE} \mathbf{j} + 5600 \mathbf{j} \\
\mathbf{a}_D &= \alpha_{BD} \mathbf{r}_B = -17 \omega_{DE} \mathbf{i} - 17 \omega_{DE} \mathbf{j} + 5600 \mathbf{j}
\end{align*}
\]

Substituting into Eq. (1) and equating the coefficients of \( \mathbf{i} \) and \( \mathbf{j} \), we obtain

\[
\begin{align*}
-17 \alpha_{DE} + 3 \alpha_{BD} &= -15,690 \\
-17 \alpha_{DE} - 12 \alpha_{BD} &= -6,010 \\
\alpha_{BD} &= -(645 \text{ rad/s}^3) \mathbf{k} & \alpha_{DE} &= (809 \text{ rad/s}^3) \mathbf{k}
\end{align*}
\]
SOLVING PROBLEMS ON YOUR OWN

This lesson was devoted to the determination of the accelerations of the points of a rigid body in plane motion. As you did previously for velocities, you will again consider the plane motion of a rigid body as the sum of two motions, namely, a translation and a rotation.

To solve a problem involving accelerations in plane motion you should use the following steps:

1. **Determine the angular velocity of the body.** To find \( \omega \) you can either
   a. Consider the motion of the body as the sum of a translation and a rotation as you did in Sec. 15.6, or
   b. Use the instantaneous center of rotation of the body as you did in Sec. 15.7. However, keep in mind that you cannot use the instantaneous center to determine accelerations.

2. **Start drawing a “diagram equation”** to use in your solution. This “equation” will involve the following diagrams (Fig. 15.44).
   a. **Plane motion diagram.** Draw a sketch of the body, including all dimensions, as well as the angular velocity \( \omega \). Show the angular acceleration \( \alpha \) with its magnitude and sense if you know them. Also show those points for which you know or seek the accelerations, indicating all that you know about these accelerations.
   b. **Translation diagram.** Select a reference point \( A \) for which you know the direction, the magnitude, or a component of the acceleration \( \mathbf{a}_A \). Draw a second diagram showing the body in translation with each point having the same acceleration as point \( A \).
   c. **Rotation diagram.** Considering point \( A \) as a fixed reference point, draw a third diagram showing the body in rotation about \( A \). Indicate the normal and tangential components of the relative accelerations of other points, such as the components \( (\mathbf{a}_{B/A})_n \) and \( (\mathbf{a}_{B/A})_t \) of the acceleration of point \( B \) with respect to point \( A \).

3. **Write the relative-acceleration formula**

   \[
   \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \quad \text{or} \quad \mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_n + (\mathbf{a}_{B/A})_t
   \]

   The sample problems illustrate three different ways to use this vector equation:
   a. **If \( \alpha \) is given or can easily be determined**, you can use this equation to determine the accelerations of various points of the body [Sample Prob. 15.6].

   (continued)
b. If \( \alpha \) cannot easily be determined, select for point \( B \) a point for which you know the direction, the magnitude, or a component of the acceleration \( \mathbf{a}_B \) and draw a vector diagram of the equation. Starting at the same point, draw all known acceleration components in tip-to-tail fashion for each member of the equation. Complete the diagram by drawing the two remaining vectors in appropriate directions and in such a way that the two sums of vectors end at a common point.

The magnitudes of the two remaining vectors can be found either graphically or analytically. Usually an analytic solution will require the solution of two simultaneous equations [Sample Prob. 15.7]. However, by first considering the components of the various vectors in a direction perpendicular to one of the unknown vectors, you may be able to obtain an equation in a single unknown.

One of the two vectors obtained by the method just described will be \( (\mathbf{a}_{BA})_t \), from which you can compute \( \alpha \). Once \( \alpha \) has been found, the vector equation can be used to determine the acceleration of any other point of the body.

c. A full vector approach can also be used to solve the vector equation. This is illustrated in Sample Prob. 15.8.

4. The analysis of plane motion in terms of a parameter completed this lesson. This method should be used only if it is possible to express the coordinates \( x \) and \( y \) of all significant points of the body in terms of a single parameter (Sec. 15.9). By differentiating twice with respect to \( t \) the coordinates \( x \) and \( y \) of a given point, you can determine the rectangular components of the absolute velocity and absolute acceleration of that point.
PROBLEMS

15.105 A 900-mm rod rests on a horizontal table. A force $P$ applied as shown produces the following accelerations: $a_A = 3.6 \text{ m/s}^2$ to the right, $\alpha = 6 \text{ rad/s}^2$ counterclockwise as viewed from above. Determine the acceleration $(a)$ of point $G$, $(b)$ of point $B$.

![Fig. P15.105 and P15.106](image)

15.106 In Prob. 15.105, determine the point of the rod that $(a)$ has no acceleration, $(b)$ has an acceleration of $2.4 \text{ m/s}^2$ to the right.

15.107 A 10-ft steel beam is lowered by means of two cables unwinding at the same speed from overhead cranes. As the beam approaches the ground, the crane operators apply brakes to slow down the unwinding motion. At the instant considered the deceleration of the cable attached at $A$ is $12 \text{ ft/s}^2$, while that of the cable at $B$ is $5 \text{ ft/s}^2$. Determine $(a)$ the angular acceleration of the beam, $(b)$ the acceleration of point $C$.

![Fig. P15.107 and P15.108](image)

15.108 The acceleration of point $C$ is $1 \text{ ft/s}^2$ downward and the angular acceleration of the beam is $0.8 \text{ rad/s}^2$ clockwise. Knowing that the angular velocity of the beam is zero at the instant considered, determine the acceleration of each cable.

15.109 and 15.110 Bar $BDE$ is attached to two links $AB$ and $CD$. Knowing that at the instant shown link $AB$ has zero angular acceleration and an angular velocity of $3 \text{ rad/s}$ clockwise, determine the acceleration $(a)$ of point $D$, $(b)$ of point $E$.

![Fig. P15.109](image) ![Fig. P15.110](image)
15.111 An automobile travels to the left at a constant speed of 48 mi/h. Knowing that the diameter of the wheel is 22 in., determine the acceleration of point C, of point D.

![Fig. P15.111](image)

15.112 A carriage is supported by a caster A and a cylinder B, each of 50-mm diameter. Knowing that at the instant shown the carriage has an acceleration of 2.4 m/s² and a velocity of 1.5 m/s, both directed to the left, determine (a) the angular accelerations of the caster and of the cylinder, (b) the accelerations of the centers of the caster and of the cylinder.

![Fig. P15.112](image)

15.113 The motion of the 75-mm-radius cylinder is controlled by the cord shown. Knowing that end E of the cord has a velocity of 300 mm/s and an acceleration of 480 mm/s², both directed upward, determine the acceleration of point A, of point B.

![Fig. P15.113 and P15.114](image)

15.114 The motion of the 75-mm-radius cylinder is controlled by the cord shown. Knowing that end E of the cord has a velocity of 300 mm/s and an acceleration of 480 mm/s², both directed upward, determine the accelerations of points C and D of the cylinder.

15.115 and 15.116 A 3-in.-radius drum is rigidly attached to a 5-in.-radius drum as shown. One of the drums rolls without sliding on the surface shown, and a cord is wound around the other drum. Knowing that at the instant shown end D of the cord has a velocity of 8 in./s and an acceleration of 30 in./s², both directed to the left, determine the accelerations of points A, B, and C of the drums.

![Fig. P15.115](image)  ![Fig. P15.116](image)
15.117 The 150-mm-radius drum rolls without slipping on a belt that moves to the left with a constant velocity of 300 mm/s. At an instant when the velocity and acceleration of the center D of the drum are as shown, determine the accelerations of points A, B, and C of the drum.

Fig. P15.117

15.118 The 18-in.-radius flywheel is rigidly attached to a 1.5-in.-radius shaft that can roll along parallel rails. Knowing that at the instant shown the center of the shaft has a velocity of 1.2 in./s and an acceleration of 0.5 in./s², both directed down to the left, determine the acceleration (a) of point A, (b) of point B.

Fig. P15.118

15.119 In the planetary gear system shown the radius of gears A, B, C, and D is 3 in. and the radius of the outer gear E is 9 in. Knowing that gear A has a constant angular velocity of 150 rpm clockwise and that the outer gear E is stationary, determine the magnitude of the acceleration of the tooth of gear D that is in contact with (a) gear A, (b) gear E.

Fig. P15.119

15.120 The disk shown has a constant angular velocity of 500 rpm counterclockwise. Knowing that rod BD is 250 mm long, determine the acceleration of collar D when (a) θ = 90°, (b) θ = 180°.

Fig. P15.120

15.121 In the two-cylinder air compressor shown the connecting rods BD and BE are each 190 mm long and crank AB rotates about the fixed point A with a constant angular velocity of 1500 rpm clockwise. Determine the acceleration of each piston when θ = 0.

Fig. P15.121
15.122 Arm AB has a constant angular velocity of 16 rad/s counterclockwise. At the instant when $\theta = 0$, determine the acceleration (a) of collar D, (b) of the midpoint G of bar BD.

Fig. P15.122, P15.123, and P15.124

15.123 Arm AB has a constant angular velocity of 16 rad/s counterclockwise. At the instant when $\theta = 90^\circ$, determine the acceleration (a) of collar D, (b) of the midpoint G of bar BD.

15.124 Arm AB has a constant angular velocity of 16 rad/s counterclockwise. At the instant when $\theta = 60^\circ$, determine the acceleration of collar D.

15.125 Knowing that crank AB rotates about point A with a constant angular velocity of 900 rpm clockwise, determine the acceleration of the piston P when $\theta = 60^\circ$.

15.126 Knowing that crank AB rotates about point A with a constant angular velocity of 900 rpm clockwise, determine the acceleration of the piston P when $\theta = 120^\circ$.

15.127 Knowing that at the instant shown rod AB has zero angular acceleration and an angular velocity of 15 rad/s counterclockwise, determine (a) the angular acceleration of arm DE, (b) the acceleration of point D.

Fig. P15.127 and P15.128

15.128 Knowing that at the instant shown rod AB has zero angular acceleration and an angular velocity of 15 rad/s counterclockwise, determine (a) the angular acceleration of member BD, (b) the acceleration of point G.

15.129 Knowing that at the instant shown rod AB has a constant angular velocity of 6 rad/s clockwise, determine the acceleration of point D.

15.130 Knowing that at the instant shown rod AB has a constant angular velocity of 6 rad/s clockwise, determine (a) the angular acceleration of member BDE, (b) the acceleration of point E.
15.131 Knowing that at the instant shown rod AB has zero angular acceleration and an angular velocity $\omega_0$ clockwise, determine (a) the angular acceleration of arm DE, (b) the acceleration of point D.

15.132 At the instant shown rod AB has zero angular acceleration and an angular velocity of 8 rad/s clockwise. Knowing that $l = 0.3$ m, determine the acceleration of the midpoint C of member BD.

15.133 and 15.134 Knowing that at the instant shown bar AB has a constant angular velocity of 4 rad/s clockwise, determine the angular acceleration (a) of bar BD, (b) of bar DE.

15.135 and 15.136 Knowing that at the instant shown bar AB has an angular velocity of 4 rad/s and an angular acceleration of 2 rad/s$^2$, both clockwise, determine the angular acceleration (a) of bar BD, (b) of bar DE by using the vector approach as is done in Sample Prob. 15.8.

15.137 Denoting by $\mathbf{r}_A$ the position vector of a point A of a rigid slab that is in plane motion, show that (a) the position vector $\mathbf{r}_C$ of the instantaneous center of rotation is

$$\mathbf{r}_C = \mathbf{r}_A + \frac{\mathbf{\omega} \times \mathbf{v}_A}{\omega^2}$$

Where $\mathbf{\omega}$ is the angular velocity of the slab and $\mathbf{v}_A$ is the velocity of point A, (b) the acceleration of the instantaneous center of rotation is zero if, and only if,

$$\mathbf{a}_A = \frac{\alpha}{\omega} \mathbf{v}_A + \mathbf{\omega} \times \mathbf{v}_A$$

where $\alpha = \dot{\omega} \mathbf{k}$ is the angular acceleration of the slab.
*15.138 The wheels attached to the ends of rod AB roll along the surfaces shown. Using the method of Sec. 15.9, derive an expression for the angular velocity of the rod in terms of \( v_B \), \( \theta \), \( l \), and \( \beta \).

*15.139 The wheels attached to the ends of rod AB roll along the surfaces shown. Using the method of Sec. 15.9 and knowing that the acceleration of wheel B is zero, derive an expression for the angular acceleration of the rod in terms of \( v_B \), \( \theta \), \( l \), and \( \beta \).

*15.140 The drive disk of the Scotch crosshead mechanism shown has an angular velocity \( \omega \) and an angular acceleration \( \alpha \), both directed counterclockwise. Using the method of Sec. 15.9, derive expressions for the velocity and acceleration of point B.

*15.141 Rod AB moves over a small wheel at C while end A moves to the right with a constant velocity \( v_A \). Using the method of Sec. 15.9, derive expressions for the angular velocity and angular acceleration of the rod.

*15.142 Rod AB moves over a small wheel at C while end A moves to the right with a constant velocity \( v_A \). Using the method of Sec. 15.9, derive expressions for the horizontal and vertical components of the velocity of point B.

*15.143 A disk of radius \( r \) rolls to the right with a constant velocity \( v \). Denoting by \( P \) the point of the rim in contact with the ground at \( t = 0 \), derive expressions for the horizontal and vertical components of the velocity of \( P \) at any time \( t \).
15.144 At the instant shown, rod \( AB \) rotates with an angular velocity \( \omega \) and an angular acceleration \( \alpha \), both clockwise. Using the method of Sec. 15.9, derive expressions for the velocity and acceleration of point \( C \).

![Fig. P15.144 and P15.145](image)

15.145 At the instant shown, rod \( AB \) rotates with an angular velocity \( \omega \) and an angular acceleration \( \alpha \), both clockwise. Using the method of Sec. 15.9, derive expressions for the horizontal and vertical components of the velocity and acceleration of point \( D \).

15.146 The position of rod \( AB \) is controlled by a disk of radius \( r \) which is attached to yoke \( CD \). Knowing that the yoke moves vertically upward with a constant velocity \( v_0 \), derive an expression for the angular acceleration of rod \( AB \).

15.147 In Prob. 15.146, derive an expression for the angular acceleration of rod \( AB \).

15.148 A wheel of radius \( r \) rolls without slipping along the inside of a fixed cylinder of radius \( R \) with a constant angular velocity \( \omega \). Denoting by \( P \) the point of the wheel in contact with the cylinder at \( t = 0 \), derive expressions for the horizontal and vertical components of the velocity of \( P \) at any time \( t \). (The curve described by point \( P \) is a hypocycloid.)

15.149 In Prob. 15.148, show that the path of \( P \) is a vertical straight line when \( r = R/2 \). Derive expressions for the corresponding velocity and acceleration of \( P \) at any time \( t \).

15.10 RATE OF CHANGE OF A VECTOR WITH RESPECT TO A ROTATING FRAME

We saw in Sec. 11.10 that the rate of change of a vector is the same with respect to a fixed frame and with respect to a frame in translation. In this section, the rates of change of a vector \( Q \) with respect to a fixed frame and with respect to a rotating frame of reference will be considered.† You will learn to determine the rate of change of \( Q \) with respect to one frame of reference when \( Q \) is defined by its components in another frame.

†It is recalled that the selection of a fixed frame of reference is arbitrary. Any frame may be designated as “fixed”; all others will then be considered as moving.

Photo 15.7 A geneva mechanism is used to convert rotary motion into intermittent motion.
Consider two frames of reference centered at $O$, a fixed frame $OXYZ$ and a frame $Oxyz$ which rotates about the fixed axis $OA$; let $\Omega$ denote the angular velocity of the frame $Oxyz$ at a given instant (Fig. 15.26). Consider now a vector function $\mathbf{Q}(t)$ represented by the vector $\mathbf{Q}$ attached at $O$; as the time $t$ varies, both the direction and the magnitude of $\mathbf{Q}$ change. Since the variation of $\mathbf{Q}$ is viewed differently by an observer using $OXYZ$ as a frame of reference and by an observer using $Oxyz$, we should expect the rate of change of $\mathbf{Q}$ to depend upon the frame of reference which has been selected. Therefore, the rate of change of $\mathbf{Q}$ with respect to the fixed frame $OXYZ$ will be denoted by $(\dot{\mathbf{Q}})_{OXYZ}$, and the rate of change of $\mathbf{Q}$ with respect to the rotating frame $Oxyz$ will be denoted by $(\dot{\mathbf{Q}})_{Oxyz}$. We propose to determine the relation existing between these two rates of change.

Let us first resolve the vector $\mathbf{Q}$ into components along the $x$, $y$, and $z$ axes of the rotating frame. Denoting by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ the corresponding unit vectors, we write

$$\mathbf{Q} = Q_x \mathbf{i} + Q_y \mathbf{j} + Q_z \mathbf{k} \quad (15.27)$$

Differentiating (15.27) with respect to $t$ and considering the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as fixed, we obtain the rate of change of $\mathbf{Q}$ with respect to the rotating frame $Oxyz$:

$$(\dot{\mathbf{Q}})_{Oxyz} = \dot{Q}_x \mathbf{i} + \dot{Q}_y \mathbf{j} + \dot{Q}_z \mathbf{k} \quad (15.28)$$

To obtain the rate of change of $\mathbf{Q}$ with respect to the fixed frame $OXYZ$, we must consider the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as variable when differentiating (15.27). We therefore write

$$(\dot{\mathbf{Q}})_{OXYZ} = \dot{Q}_x \mathbf{i} + \dot{Q}_y \mathbf{j} + \dot{Q}_z \mathbf{k} + \frac{d\mathbf{i}}{dt} + \frac{d\mathbf{j}}{dt} + \frac{d\mathbf{k}}{dt} \quad (15.29)$$

Recalling (15.28), we observe that the sum of the first three terms in the right-hand member of (15.29) represents the rate of change $(\dot{\mathbf{Q}})_{Oxyz}$. We note, on the other hand, that the rate of change $(\dot{\mathbf{Q}})_{OXYZ}$ would reduce to the last three terms in (15.29) if the vector $\mathbf{Q}$ were fixed within the frame $Oxyz$, since $(\dot{\mathbf{Q}})_{Oxyz}$ would then be zero. But in that case, $(\dot{\mathbf{Q}})_{Oxyz}$ would represent the velocity of a particle located at the tip of $\mathbf{Q}$ and belonging to a body rigidly attached to the frame $Oxyz$. Thus, the last three terms in (15.29) represent the velocity of that particle; since the frame $Oxyz$ has an angular velocity $\Omega$ with respect to $OXYZ$ at the instant considered, we write, by (15.5),

$$Q_x \frac{d\mathbf{i}}{dt} + Q_y \frac{d\mathbf{j}}{dt} + Q_z \frac{d\mathbf{k}}{dt} = \Omega \times \mathbf{Q} \quad (15.30)$$

Substituting from (15.28) and (15.30) into (15.29), we obtain the fundamental relation

$$(\dot{\mathbf{Q}})_{OXYZ} = (\dot{\mathbf{Q}})_{Oxyz} + \Omega \times \mathbf{Q} \quad (15.31)$$

We conclude that the rate of change of the vector $\mathbf{Q}$ with respect to the fixed frame $OXYZ$ is made of two parts: The first part represents the rate of change of $\mathbf{Q}$ with respect to the rotating frame $Oxyz$; the second part, $\Omega \times \mathbf{Q}$, is induced by the rotation of the frame $Oxyz$. 

---

**Fig. 15.26**
The use of relation (15.31) simplifies the determination of the rate of change of a vector $Q$ with respect to a fixed frame of reference $OXYZ$ when the vector $Q$ is defined by its components along the axes of a rotating frame $Oxyz$, since this relation does not require the separate computation of the derivatives of the unit vectors defining the orientation of the rotating frame.

15.11 PLANE MOTION OF A PARTICLE RELATIVE TO A ROTATING FRAME. CORIOLIS ACCELERATION

Consider two frames of reference, both centered at $O$ and both in the plane of the figure, a fixed frame $OXY$ and a rotating frame $Oxy$ (Fig. 15.27). Let $P$ be a particle moving in the plane of the figure. The position vector $r$ of $P$ is the same in both frames, but its rate of change depends upon the frame of reference which has been selected.

The absolute velocity $v_P$ of the particle is defined as the velocity observed from the fixed frame $OXY$ and is equal to the rate of change $(\dot{r})_{OXY}$ of $r$ with respect to that frame. We can, however, express $v_P$ in terms of the rate of change $(\dot{r})_{Oxy}$ observed from the rotating frame if we make use of Eq. (15.31). Denoting by $\Omega$ the angular velocity of the frame $Oxy$ with respect to $OXY$ at the instant considered, we write

$$ v_P = (\dot{r})_{OXY} = \Omega \times r + (\dot{r})_{Oxy} \quad (15.32) $$

But $(\dot{r})_{Oxy}$ defines the velocity of the particle $P$ relative to the rotating frame $Oxy$. Denoting the rotating frame by $F$ for short, we represent the velocity $(\dot{r})_{Oxy}$ of $P$ relative to the rotating frame by $v_{P/F}$. Let us imagine that a rigid slab has been attached to the rotating frame. Then $v_{P/F}$ represents the velocity of $P$ along the path that it describes on that slab (Fig. 15.28), and the term $\Omega \times r$ in (15.32) represents the velocity $v_P$ of the point $P'$ of the slab—or rotating frame—which coincides with $P$ at the instant considered. Thus, we have

$$ v_P = v_{P'} + v_{P/F} \quad (15.33) $$

where $v_P =$ absolute velocity of particle $P$
$v_{P'} =$ velocity of point $P'$ of moving frame $F$ coinciding with $P$
$v_{P/F} =$ velocity of $P$ relative to moving frame $F$

The absolute acceleration $a_P$ of the particle is defined as the rate of change of $v_P$ with respect to the fixed frame $OXY$. Computing the rates of change with respect to $OXY$ of the terms in (15.32), we write

$$ a_P = \dot{v}_P = \dot{\Omega} \times r + \Omega \times \dot{r} + \frac{d}{dt}[(\dot{r})_{Oxy}] \quad (15.34) $$

where all derivatives are defined with respect to $OXY$, except where indicated otherwise. Referring to Eq. (15.31), we note that the last term in (15.34) can be expressed as

$$ \frac{d}{dt}[(\dot{r})_{Oxy}] = (\dot{r})_{Oxy} + \Omega \times (\dot{r})_{Oxy} $$
On the other hand, \( \dot{\mathbf{r}} \) represents the velocity \( \mathbf{v}_P \) and can be replaced by the right-hand member of Eq. (15.32). After completing these two substitutions into (15.34), we write

\[
a_P = \dot{\Omega} \times \mathbf{r} + \Omega \times (\Omega \times \mathbf{r}) + 2\Omega \times (\hat{\mathbf{r}})_{Oxy} + (\mathbf{r})_{Oxy} (15.35)
\]

Referring to the expression (15.8) obtained in Sec. 15.3 for the acceleration of a particle in a rigid body rotating about a fixed axis, we note that the sum of the first two terms represents the acceleration \( \mathbf{a}_P \) of the point \( P' \) of the rotating frame which coincides with \( P \) at the instant considered. On the other hand, the last term defines the acceleration \( \mathbf{a}_{P/\mathcal{F}} \) of \( P \) relative to the rotating frame. If it were not for the third term, which has not been accounted for, a relation similar to (15.33) could be written for the accelerations, and \( \mathbf{a}_P \) could be expressed as the sum of \( \mathbf{a}_P \) and \( \mathbf{a}_{P/\mathcal{F}} \). However, it is clear that such a relation would be incorrect and that we must include the additional term. This term, which will be denoted by \( \mathbf{a}_c \), is called the complementary acceleration, or Coriolis acceleration, after the French mathematician de Coriolis (1792–1843). We write

\[
\mathbf{a}_P = \mathbf{a}_P + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c (15.36)
\]

where \( \mathbf{a}_P \) = absolute acceleration of particle \( P \)
\( \mathbf{a}_P \) = acceleration of point \( P' \) of moving frame \( \mathcal{F} \) coinciding with \( P \)
\( \mathbf{a}_{P/\mathcal{F}} \) = acceleration of \( P \) relative to moving frame \( \mathcal{F} \)
\( \mathbf{a}_c = 2\Omega \times (\hat{\mathbf{r}})_{Oxy} = 2\Omega \times \mathbf{v}_{P/\mathcal{F}} 
\]

\( = \) complementary, or Coriolis, acceleration†

We note that since point \( P' \) moves in a circle about the origin \( O \), its acceleration \( \mathbf{a}_P \) has, in general, two components: a component \( \langle \mathbf{a}_P \rangle_h \) tangent to the circle, and a component \( \langle \mathbf{a}_P \rangle_n \) directed toward \( O \). Similarly, the acceleration \( \mathbf{a}_{P/\mathcal{F}} \) generally has two components: a component \( \langle \mathbf{a}_{P/\mathcal{F}} \rangle_h \) tangent to the path that \( P \) describes on the rotating slab, and a component \( \langle \mathbf{a}_{P/\mathcal{F}} \rangle_n \) directed toward the center of curvature of that path. We further note that since the vector \( \Omega \) is perpendicular to the plane of motion, and thus to \( \mathbf{v}_{P/\mathcal{F}} \), the magnitude of the Coriolis acceleration \( \mathbf{a}_c = 2\Omega \times \mathbf{v}_{P/\mathcal{F}} \) is equal to \( 2\Omega\mathbf{v}_{P/\mathcal{F}} \), and its direction can be obtained by rotating the vector \( \mathbf{v}_{P/\mathcal{F}} \) through 90° in the sense of rotation of the moving frame (Fig. 15.29). The Coriolis acceleration reduces to zero when either \( \Omega \) or \( \mathbf{v}_{P/\mathcal{F}} \) is zero.

The following example will help in understanding the physical meaning of the Coriolis acceleration. Consider a collar \( P \) which is

†It is important to note the difference between Eq. (15.36) and Eq. (15.21) of Sec. 15.8. When we wrote

\[
\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA} (15.21)
\]

in Sec. 15.8, we were expressing the absolute acceleration of point \( B \) as the sum of its acceleration \( \mathbf{a}_{BA} \) relative to a frame in translation and of the acceleration \( \mathbf{a}_A \) of a point of that frame. We are now trying to relate the absolute acceleration of point \( P \) to its acceleration \( \mathbf{a}_{P/\mathcal{F}} \) relative to a rotating frame \( \mathcal{F} \) and to the acceleration \( \mathbf{a}_P \) of the point \( P' \) of that frame which coincides with \( P \). Eq. (15.36) shows that because the frame is rotating, it is necessary to include an additional term representing the Coriolis acceleration \( \mathbf{a}_c \).
made to slide at a constant relative speed \( u \) along a rod \( OB \) rotating at a constant angular velocity \( \omega \) about \( O \) (Fig. 15.30a). According to formula (15.36), the absolute acceleration of \( P \) can be obtained by adding vectorially the acceleration \( \mathbf{a}_A \) of the point \( A \) of the rod coinciding with \( P \), the relative acceleration \( \mathbf{a}_{P/OB} \) of \( P \) with respect to the rod, and the Coriolis acceleration \( \mathbf{a}_c \). Since the angular velocity \( \omega \) of the rod is constant, \( \mathbf{a}_A \) reduces to its normal component \( (\mathbf{a}_A)_n \) of magnitude \( r\omega^2 \); and since \( u \) is constant, the relative acceleration \( \mathbf{a}_{P/OB} \) is zero. According to the definition given above, the Coriolis acceleration is a vector perpendicular to \( OB \), of magnitude \( 2\omega u \), and directed as shown in the figure. The acceleration of the collar \( P \) consists, therefore, of the two vectors shown in Fig. 15.30a. Note that the result obtained can be checked by applying the relation (11.44).

To understand better the significance of the Coriolis acceleration, let us consider the absolute velocity of \( P \) at time \( t \) and at time \( t + \Delta t \) (Fig. 15.30b). The velocity at time \( t \) can be resolved into its components \( \mathbf{u} \) and \( \mathbf{v}_A \); the velocity at time \( t + \Delta t \) can be resolved into its components \( \mathbf{u}' \) and \( \mathbf{v}_A' \). Drawing these components from the same origin (Fig. 15.30c), we note that the change in velocity during the time \( \Delta t \) can be represented by the sum of three vectors, \( \Delta \mathbf{r} \), \( \Delta \mathbf{v} \), and \( \Delta \mathbf{a} \). The vector \( \Delta \mathbf{r} \) measures the change in direction of the velocity \( \mathbf{v}_A \), and the quotient \( \Delta \mathbf{v} / \Delta t \) represents the acceleration \( \mathbf{a}_A \) when \( \Delta t \) approaches zero. We check that the direction of \( \Delta \mathbf{v} / \Delta t \) is that of \( \mathbf{a}_A \) when \( \Delta t \) approaches zero and that

\[
\lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \lim_{\Delta t \to 0} \mathbf{v}_A \frac{\Delta t}{\Delta t} = 2\mathbf{v}_A = r\omega = 2\omega^2 = \mathbf{a}_A
\]

The vector \( \Delta \mathbf{r} \) measures the change in direction of \( \mathbf{u} \) due to the rotation of the rod; the vector \( \Delta \mathbf{v} \) measures the change in magnitude of \( \mathbf{v}_A \) due to the motion of \( P \) on the rod. The vectors \( \Delta \mathbf{r} \) and \( \Delta \mathbf{v} \) result from the combined effect of the relative motion of \( P \) and of the rotation of the rod; they would vanish if either of these two motions stopped. It is easily verified that the sum of these two vectors defines the Coriolis acceleration. Their direction is that of \( \mathbf{v}_A \) and at time \( t \) at time \( t + \Delta t \). It is obtained by adding vectorially the acceleration

\[
\lim_{\Delta t \to 0} \left( \frac{\Delta \mathbf{r}}{\Delta t} + \frac{\Delta \mathbf{v}}{\Delta t} \right) = \lim_{\Delta t \to 0} \left( \frac{\Delta \mathbf{r}}{\Delta t} + \frac{\Delta \mathbf{r}}{\Delta t} \right) = u\omega + \omega u + 2\omega u
\]

Formulas (15.33) and (15.36) can be used to analyze the motion of mechanisms which contain parts sliding on each other. They make it possible, for example, to relate the absolute and relative motions of sliding pins and collars (see Sample Probs. 15.9 and 15.10). The concept of Coriolis acceleration is also very useful in the study of long-range projectiles and of other bodies whose motions are appreciably affected by the rotation of the earth. As was pointed out in Sec. 12.2, a system of axes attached to the earth does not truly constitute a newtonian frame of reference; such a system of axes should actually be considered as rotating. The formulas derived in this section will therefore facilitate the study of the motion of bodies with respect to axes attached to the earth.
SAMPLE PROBLEM 15.9

The Geneva mechanism shown is used in many counting instruments and in other applications where an intermittent rotary motion is required. Disk $D$ rotates with a constant counterclockwise angular velocity $\omega_D$ of 10 rad/s. A pin $P$ is attached to disk $D$ and slides along one of several slots cut in disk $S$. It is desirable that the angular velocity of disk $S$ be zero as the pin enters and leaves each slot; in the case of four slots, this will occur if the distance between the centers of the disks is $l = \sqrt{2} R$.

At the instant when $\phi = 150^\circ$, determine (a) the angular velocity of disk $S$, (b) the velocity of pin $P$ relative to disk $S$.

\[ 976 \]

\[ \text{Solution} \]

We solve triangle $OPB$, which corresponds to the position $\phi = 150^\circ$. Using the law of cosines, we write

\[ r^2 = R^2 + l^2 - 2RL \cos 30^\circ = 0.551R^2 \quad r = 0.742R = 37.1 \text{ mm} \]

From the law of sines,

\[ \frac{\sin \beta}{R} = \frac{\sin 30^\circ}{r} \quad \beta = 42.4^\circ \]

Since pin $P$ is attached to disk $D$, and since disk $D$ rotates about point $B$, the magnitude of the absolute velocity of $P$ is

\[ v_P = R\omega_D = (50 \text{ mm})(10 \text{ rad/s}) = 500 \text{ mm/s} \quad v_P = 500 \text{ mm/s} \uparrow 60^\circ \]

We consider now the motion of pin $P$ along the slot in disk $S$. Denoting by $P'$ the point of disk $S$ which coincides with $P$ at the instant considered and selecting a rotating frame $S'$ attached to disk $S$, we write

\[ v_P = v_{P'} + v_{P/S} \]

Noting that $v_{P'}$ is perpendicular to the radius $OP$ and that $v_{P/S}$ is directed along the slot, we draw the velocity triangle corresponding to the equation above. From the triangle, we compute

\[ \gamma = 90^\circ - 42.4^\circ - 30^\circ = 17.6^\circ \]
\[ v_{P'} = v_P \sin \gamma = (500 \text{ mm/s}) \sin 17.6^\circ \]
\[ v_{P'} = 151.2 \text{ mm/s} \uparrow 42.4^\circ \]
\[ v_{P/S} = v_{P/S} = 477 \text{ mm/s} \uparrow 42.4^\circ \]

Since $v_P$ is perpendicular to the radius $OP$, we write

\[ v_P = r\omega_S \quad 151.2 \text{ mm/s} = (37.1 \text{ mm})\omega_S \quad \omega_S = \omega_D = 4.08 \text{ rad/s} \downarrow \]
SAMPLE PROBLEM 15.10

In the Geneva mechanism of Sample Prob. 15.9, disk $D$ rotates with a constant counterclockwise angular velocity $\omega_D$ of 10 rad/s. At the instant when $\phi = 150^\circ$, determine the angular acceleration of disk $S$.

SOLUTION

Referring to Sample Prob. 15.9, we obtain the angular velocity of the frame $S$ attached to disk $S$ and the velocity of the pin relative to $S$:

$\omega_S = 4.08$ rad/s $\perp$

$\beta = 42.4^\circ$

$v_{PS} = 477$ mm/s $\perp 42.4^\circ$

Since pin $P$ moves with respect to the rotating frame $S$, we write

$\mathbf{a}_P = \mathbf{a}_P^o + \mathbf{a}_{PS} + \mathbf{a}_c$ (1)

Each term of this vector equation is investigated separately.

Absolute Acceleration $\mathbf{a}_P$. Since disk $D$ rotates with a constant angular velocity, the absolute acceleration $\mathbf{a}_P$ is directed toward $B$. We have

$\mathbf{a}_P = R\mathbf{\omega}_D^2 = (500 \text{ mm})(10 \text{ rad/s})^2 = 5000 \text{ mm/s}^2$

$\mathbf{a}_P = 5000 \text{ mm/s}^2 \perp 30^\circ$

Acceleration $\mathbf{a}_{P^o}$ of the Coinciding Point $P'$. The acceleration $\mathbf{a}_{P^o}$ of the point $P'$ of the frame $S$ which coincides with $P$ at the instant considered is resolved into normal and tangential components. (We recall from Sample Prob. 15.9 that $r = 37.1$ mm.)

$(\mathbf{a}_{P^o})_n = r\omega_S^2 = (37.1 \text{ mm})(4.08 \text{ rad/s})^2 = 618 \text{ mm/s}^2$

$(\mathbf{a}_{P^o})_t = 618 \text{ mm/s}^2 \perp 42.4^\circ$

$(\mathbf{a}_S)_l = r\alpha_S = 37.1\alpha_S \perp 42.4^\circ$

Relative Acceleration $\mathbf{a}_{PS}$. Since the pin $P$ moves in a straight slot cut in disk $S$, the relative acceleration $\mathbf{a}_{PS}$ must be parallel to the slot; i.e., its direction must be $\perp 42.4^\circ$.

Coriolis Acceleration $\mathbf{a}_c$. Rotating the relative velocity $v_{PS}$ through 90° in the sense of $\omega_S$, we obtain the direction of the Coriolis component of the acceleration: $\perp 42.4^\circ$. We write

$\mathbf{a}_c = 2\omega_S v_{PS} = 2(4.08 \text{ rad/s})(477 \text{ mm/s}) = 3890 \text{ mm/s}^2$

$\mathbf{a}_c = 3890 \text{ mm/s}^2 \perp 42.4^\circ$

We rewrite Eq. (1) and substitute the accelerations found above:

$\mathbf{a}_P = (\mathbf{a}_{P^o})_n + (\mathbf{a}_{P^o})_t + \mathbf{a}_{PS} + \mathbf{a}_c$

$[5000 \perp 30^\circ] = [618 \perp 42.4^\circ] + [37.1\alpha_S \perp 42.4^\circ]$

Equating components in a direction perpendicular to the slot,

$5000 \cos 17.6^\circ = 37.1\alpha_S - 3890$

$\alpha_S = \alpha_S = 233 \text{ rad/s}^2$
In this lesson you studied the rate of change of a vector with respect to a rotating frame and then applied your knowledge to the analysis of the plane motion of a particle relative to a rotating frame.

1. Rate of change of a vector with respect to a fixed frame and with respect to a rotating frame. Denoting by \( \dot{Q}_{OXYZ} \) the rate of change of a vector \( Q \) with respect to a fixed frame \( OXYZ \) and by \( \dot{Q}_{Oxyz} \) its rate of change with respect to a rotating frame \( Oxyz \), we obtained the fundamental relation

\[
(\dot{Q}_{OXYZ}) = (\dot{Q}_{Oxyz}) + \Omega \times Q
\]

(15.31)

where \( \Omega \) is the angular velocity of the rotating frame.

This fundamental relation will now be applied to the solution of two-dimensional problems.

2. Plane motion of a particle relative to a rotating frame. Using the above fundamental relation and designating by \( \mathcal{F} \) the rotating frame, we obtained the following expressions for the velocity and the acceleration of a particle \( P \):

\[
v_P = v_{P'} + v_{P/\mathcal{F}}
\]

(15.33)

\[
a_P = a_{P'} + a_{P/\mathcal{F}} + a_c
\]

(15.36)

In these equations:

- **a. The subscript** \( P \) **refers to the absolute motion of the particle** \( P \), that is, to its motion with respect to a fixed frame of reference \( OXY \). 

- **b. The subscript** \( P' \) **refers to the motion of the point** \( P' \) **of the rotating frame** \( \mathcal{F} \) **which coincides with** \( P \) **at the instant considered.**

- **c. The subscript** \( P/\mathcal{F} \) **refers to the motion of the particle** \( P \) **relative to the rotating frame** \( \mathcal{F} \).

- **d. The term** \( a_c \) **represents the Coriolis acceleration of point** \( P \). **Its magnitude is** \( 2\Omega v_{P/\mathcal{F}} \), **and its direction is found by rotating** \( v_{P/\mathcal{F}} \) **through 90° in the sense of rotation of the frame** \( \mathcal{F} \).

You should keep in mind that the Coriolis acceleration should be taken into account whenever a part of the mechanism you are analyzing is moving with respect to another part that is rotating. The problems you will encounter in this lesson involve collars that slide on rotating rods, booms that extend from cranes rotating in a vertical plane, etc.

When solving a problem involving a rotating frame, you will find it convenient to draw vector diagrams representing Eqs. (15.33) and (15.36), respectively, and use these diagrams to obtain either an analytical or a graphical solution.
**PROBLEMS**

**15.150 and 15.151** Two rotating rods are connected by slider block $P$. The rod attached at $A$ rotates with a constant angular velocity $\omega_A$. For the given data, determine for the position shown (a) the angular velocity of the rod attached at $B$, (b) the relative velocity of slider block $P$ with respect to the rod on which it slides.

15.150 $b = 8\text{ in.}$, $\omega_A = 6\text{ rad/s}$.
15.151 $b = 300\text{ mm}$, $\omega_A = 10\text{ rad/s}$.

![Fig. P15.150 and P15.152](image)

**15.152 and 15.153** Two rotating rods are connected by slider block $P$. The velocity $v_0$ of the slider block relative to the rod is constant and is directed outwards. For the given data, determine the angular velocity of each rod in the position shown.

15.152 $b = 300\text{ mm}$, $v_0 = 480\text{ mm/s}$.
15.153 $b = 8\text{ in.}$, $v_0 = 9\text{ in/s}$.

![Fig. P15.151 and P15.153](image)

**15.154 and 15.155** Pin $P$ is attached to the collar shown; the motion of the pin is guided by a slot cut in rod $BD$ and by the collar that slides on rod $AE$. Knowing that at the instant considered the rods rotate clockwise with constant angular velocities, determine for the given data the velocity of pin $F$.

15.154 $\omega_{AE} = 4\text{ rad/s}$, $\omega_{BD} = 1.5\text{ rad/s}$.
15.155 $\omega_{AE} = 3.5\text{ rad/s}$, $\omega_{BD} = 2.4\text{ rad/s}$.

![Fig. P15.154 and P15.155](image)
15.156 and 15.157 Two rods $AE$ and $BD$ pass through holes drilled into a hexagonal block. (The holes are drilled in different planes so that the rods will not touch each other.) Knowing that at the instant considered rod $AE$ rotates counterclockwise with a constant angular velocity $\omega$, determine, for the given data, the relative velocity of the block with respect to each rod.

15.156 $(a) \theta = 90^\circ$, $(b) \theta = 60^\circ$.
15.157 $\theta = 45^\circ$.

15.158 Four pins slide in four separate slots cut in a circular plate as shown. When the plate is at rest, each pin has a velocity directed as shown and of the same constant magnitude $u$. If each pin maintains the same velocity relative to the plate when the plate rotates about $O$ with a constant counterclockwise angular velocity $\omega$, determine the acceleration of each pin.

15.159 Solve Prob. 15.158, assuming that the plate rotates about $O$ with a constant clockwise angular velocity $\omega$.

15.160 At the instant shown the length of the boom $AB$ is being decreased at the constant rate of $0.2 \text{ m/s}$ and the boom is being lowered at the constant rate of $0.08 \text{ rad/s}$. Determine $(a)$ the velocity of point $B$, $(b)$ the acceleration of point $B$.

15.161 At the instant shown the length of the boom $AB$ is being increased at the constant rate of $0.2 \text{ m/s}$ and the boom is being lowered at the constant rate of $0.08 \text{ rad/s}$. Determine $(a)$ the velocity of point $B$, $(b)$ the acceleration of point $B$.

15.162 and 15.163 The sleeve $BC$ is welded to an arm that rotates about $A$ with a constant angular velocity $\omega$. In the position shown rod $DF$ is being moved to the left at a constant speed $u = 16 \text{ in/s}$ relative to the sleeve. For the given angular velocity $\omega$, determine the acceleration $(a)$ of point $D$, $(b)$ of the point of rod $DF$ that coincides with point $E$.

15.162 $\omega = (3 \text{ rad/s}) \mathbf{i}$.
15.163 $\omega = (3 \text{ rad/s}) \mathbf{j}$.
15.164 The cage of a mine elevator moves downward at a constant speed of 40 ft/s. Determine the magnitude and direction of the Coriolis acceleration of the cage if the elevator is located (a) at the equator, (b) at latitude 40° north, (c) at latitude 40° south.

15.165 A rocket sled is tested on a straight track that is built along a meridian. Knowing that the track is located at latitude 40° north, determine the Coriolis acceleration of the sled when it is moving north at a speed of 900 km/h.

15.166 The motion of nozzle D is controlled by arm AB. At the instant shown the arm is rotating counterclockwise at the constant rate $\omega = 2.4 \text{ rad/s}$ and portion BC is being extended at the constant rate $u = 10 \text{ in./s}$ with respect to the arm. For each of the arrangements shown, determine the acceleration of the nozzle D.

15.167 Solve Prob. 15.166, assuming that the direction of the relative velocity $u$ is reversed so that portion BD is being retracted.

15.168 and 15.169 A chain is looped around two gears of radius 40 mm that can rotate freely with respect to the 320-mm arm AB. The chain moves about arm AB in a clockwise direction at the constant rate of 80 mm/s relative to the arm. Knowing that in the position shown arm AB rotates clockwise about A at the constant rate $\omega = 0.75 \text{ rad/s}$, determine the acceleration of each of the chain links indicated.

15.168 Links 1 and 2.
15.169 Links 3 and 4.

15.170 Rod AB of length R rotates about A with a constant clockwise angular velocity $\omega_1$. At the same time, rod BD of length r rotates about B with a constant counterclockwise angular velocity $\omega_2$ with respect to rod AB. Show that if $\omega_2 = 2\omega_1$, the acceleration of point D passes through point A. Further show that this result is independent of R, r, and $\theta$.

15.171 Rod AB of length $R = 15 \text{ in.}$ rotates about A with a constant clockwise angular velocity $\omega_1$ of 5 rad/s. At the same time, rod BD of length $r = 8 \text{ in.}$ rotates about B with a constant counterclockwise angular velocity $\omega_2$ of 3 rad/s with respect to rod AB. Knowing that $\theta = 60^\circ$, determine for the position shown the acceleration of point D.
15.172 The collar $P$ slides outward at a constant relative speed $u$ along rod $AB$, which rotates counterclockwise with a constant angular velocity of 20 rpm. Knowing that $r = 250$ mm when $\theta = 0$ and that the collar reaches $B$ when $\theta = 90^\circ$, determine the magnitude of the acceleration of the collar $P$ just as it reaches $B$.

![Fig. P15.172](image)

15.173 Pin $P$ slides in a circular slot cut in the plate shown at a constant relative speed $u = 90$ mm/s. Knowing that at the instant shown the plate rotates clockwise about $A$ at the constant rate $\omega = 3$ rad/s, determine the acceleration of the pin if it is located at (a) point $A$, (b) point $B$, (c) point $C$.

![Fig. P15.173 and P15.174](image)

15.174 Pin $P$ slides in a circular slot cut in the plate shown at a constant relative speed $u = 90$ mm/s. Knowing that at the instant shown the angular velocity $\omega$ of the plate is 3 rad/s clockwise and is decreasing at the rate of 5 rad/s$^2$, determine the acceleration of the pin if it is located at (a) point $A$, (b) point $B$, (c) point $C$.

15.175 and 15.176 Knowing that at the instant shown the rod attached at $B$ rotates with a constant counterclockwise angular velocity $\omega_B$ of 6 rad/s, determine the angular velocity and angular acceleration of the rod attached at $A$.

![Fig. P15.175](image)  ![Fig. P15.176](image)
15.177 At the instant shown bar BC has an angular velocity of 3 rad/s and an angular acceleration of 2 rad/s², both counterclockwise, determine the angular acceleration of the plate.

![Diagram of bar BC with labeled dimensions](image)

**Fig. P15.177 and P15.178**

15.178 At the instant shown bar BC has an angular velocity of 3 rad/s and an angular acceleration of 2 rad/s², both clockwise, determine the angular acceleration of the plate.

15.179 The Geneva mechanism shown is used to provide an intermittent rotary motion of disk S. Disk D rotates with a constant counterclockwise angular velocity $\omega_D$ of 8 rad/s. A pin $P$ is attached to disk D and can slide in one of the six equally spaced slots cut in disk S. It is desirable that the angular velocity of disk S be zero as the pin enters and leaves each of the six slots; this will occur if the distance between the centers of the disks and the radii of the disks are related as shown. Determine the angular velocity and angular acceleration of disk S at the instant when $\phi = 150^\circ$.

![Diagram of Geneva mechanism](image)

**Fig. P15.179**

15.180 In Prob. 15.179, determine the angular velocity and angular acceleration of disk S at the instant when $\phi = 135^\circ$.

15.181 The disk shown rotates with a constant clockwise angular velocity of 12 rad/s. At the instant shown, determine (a) the angular velocity and angular acceleration of rod BD, (b) the velocity and acceleration of the point of the rod coinciding with E.

![Diagram of disk rotation](image)

**Fig. P15.181**
15.182 Rod AB passes through a collar which is welded to link DE. Knowing that at the instant shown block A moves to the right at a constant speed of 75 in./s, determine (a) the angular velocity of rod AB, (b) the velocity relative to the collar of the point of the rod in contact with the collar, (c) the acceleration of the point of the rod in contact with the collar. (Hint: Rod AB and link DE have the same \( \omega \) and the same \( \alpha \).)

![Figure P15.182](image)

15.183 Solve Prob. 15.182 assuming block A moves to the left at a constant speed of 75 in./s.

15.12 MOTION ABOUT A FIXED POINT

In Sec. 15.3 the motion of a rigid body constrained to rotate about a fixed axis was considered. The more general case of the motion of a rigid body which has a fixed point \( O \) will now be examined.

First, it will be proved that the most general displacement of a rigid body with a fixed point \( O \) is equivalent to a rotation of the body about an axis through \( O \).† Instead of considering the rigid body itself, we can detach a sphere of center \( O \) from the body and analyze the motion of that sphere. Clearly, the motion of the sphere completely characterizes the motion of the given body. Since three points define the position of a solid in space, the center \( O \) and two points \( A \) and \( B \) on the surface of the sphere will define the position of the sphere and thus the position of the body. Let \( A_1 \) and \( B_1 \) characterize the position of the sphere at one instant, and let \( A_2 \) and \( B_2 \) characterize its position at a later instant (Fig. 15.31a). Since the sphere is rigid, the lengths of the arcs of great circle \( A_1B_1 \) and \( A_2B_2 \) must be equal, but except for this requirement, the positions of \( A_1, A_2, B_1, \) and \( B_2 \) are arbitrary. We propose to prove that the points \( A \) and \( B \) can be brought, respectively, from \( A_1 \) and \( B_1 \) into \( A_2 \) and \( B_2 \) by a single rotation of the sphere about an axis.

For convenience, and without loss of generality, we select point \( B \) so that its initial position coincides with the final position of \( A \); thus, \( B_1 = A_2 \) (Fig. 15.31b). We draw the arcs of great circle \( A_1A_2, A_2B_2 \) and the arcs bisecting, respectively, \( A_1A_2 \) and \( A_2B_2 \). Let \( C \) be the point of intersection of these last two arcs; we complete the construction by drawing \( A_1C, A_2C, \) and \( B_2C \). As pointed out above, because of the rigidity of the sphere, \( A_1B_1 = A_2B_2 \). Since \( C \) is by construction equidistant from \( A_1, A_2, \) and \( B_2 \), we also have \( A_1C = A_2C = B_2C \).

†This is known as Euler's theorem.
As a result, the spherical triangles $A_1CA_2$ and $B_1CB_2$ are congruent and the angles $A_1CA_2$ and $B_1CB_2$ are equal. Denoting by $\theta$ the common value of these angles, we conclude that the sphere can be brought from its initial position into its final position by a single rotation through $\theta$ about the axis $OC$.

It follows that the motion during a time interval $\Delta t$ of a rigid body with a fixed point $O$ can be considered as a rotation through $\Delta \theta$ about a certain axis. Drawing along that axis a vector of magnitude $\Delta \theta / \Delta t$ and letting $\Delta t$ approach zero, we obtain at the limit the *instantaneous axis of rotation* and the angular velocity $\omega$ of the body at the instant considered (Fig. 15.32). The velocity of a particle $P$ of the body can then be obtained, as in Sec. 15.3, by forming the vector product of $\omega$ and of the position vector $r$ of the particle:

$$v = \frac{dr}{dt} = \omega \times r \quad (15.37)$$

The acceleration of the particle is obtained by differentiating (15.37) with respect to $t$. As in Sec. 15.3 we have

$$a = \alpha \times r + \omega \times (\omega \times r) \quad (15.38)$$

where the angular acceleration $\alpha$ is defined as the derivative

$$\alpha = \frac{d\omega}{dt} \quad (15.39)$$

of the angular velocity $\omega$.

In the case of the motion of a rigid body with a fixed point, the direction of $\omega$ and of the instantaneous axis of rotation changes from one instant to the next. The angular acceleration $\alpha$ therefore reflects the change in direction of $\omega$ as well as its change in magnitude and, in general, is not directed along the *instantaneous axis of rotation*. While the particles of the body located on the instantaneous axis of rotation have zero velocity at the instant considered, they do not have zero acceleration. Also, the accelerations of the various particles of the body cannot be determined as if the body were rotating permanently about the instantaneous axis.

Recalling the definition of the velocity of a particle with position vector $r$, we note that the angular acceleration $\alpha$, as expressed in (15.39), represents the velocity of the tip of the vector $\omega$. This property may be useful in the determination of the angular acceleration of a rigid body. For example, it follows that the vector $\alpha$ is tangent to the curve described in space by the tip of the vector $\omega$.

We should note that the vector $\omega$ moves within the body, as well as in space. It thus generates two cones called, respectively, the *body cone* and the *space cone* (Fig. 15.33).† It can be shown that at any given instant, the two cones are tangent along the instantaneous axis of rotation and that as the body moves, the body cone appears to *roll* on the space cone.

†It is recalled that a *cone* is, by definition, a surface generated by a straight line passing through a fixed point. In general, the cones considered here will not be circular cones.
Before concluding our analysis of the motion of a rigid body with a fixed point, we should prove that angular velocities are actually vectors. As indicated in Sec. 2.3, some quantities, such as the finite rotations of a rigid body, have magnitude and direction but do not obey the parallelogram law of addition; these quantities cannot be considered as vectors. In contrast, angular velocities (and also infinitesimal rotations), as will be demonstrated presently, do obey the parallelogram law and thus are truly vector quantities.

Consider a rigid body with a fixed point \( O \) which at a given instant rotates simultaneously about the axes \( OA \) and \( OB \) with angular velocities \( \omega_1 \) and \( \omega_2 \) (Fig. 15.34a). We know that this motion must be equivalent at the instant considered to a single rotation of angular velocity \( \omega \). We propose to show that

\[
\omega = \omega_1 + \omega_2
\]  

(15.40)

i.e., that the resulting angular velocity can be obtained by adding \( \omega_1 \) and \( \omega_2 \) by the parallelogram law (Fig. 15.34b).

Consider a particle \( P \) of the body, defined by the position vector \( r \). Denoting, respectively, by \( v_1, v_2, \) and \( v \) the velocity of \( P \) when the body rotates about \( OA \) only, about \( OB \) only, and about both axes simultaneously, we write

\[
v = \omega \times r \quad v_1 = \omega_1 \times r \quad v_2 = \omega_2 \times r
\]  

(15.41)

But the vectorial character of linear velocities is well established (since they represent the derivatives of position vectors). We therefore have

\[
v = v_1 + v_2
\]

where the plus sign indicates vector addition. Substituting from (15.41), we write

\[
\omega \times r = \omega_1 \times r + \omega_2 \times r
\]

\[
\omega \times r = (\omega_1 + \omega_2) \times r
\]

where the plus sign still indicates vector addition. Since the relation obtained holds for an arbitrary \( r \), we conclude that (15.40) must be true.
**15.13 GENERAL MOTION**

The most general motion of a rigid body in space will now be considered. Let A and B be two particles of the body. We recall from Sec. 11.12 that the velocity of B with respect to the fixed frame of reference OXYZ can be expressed as

\[ v_B = v_A + v_{BA} \]  \hspace{1cm} (15.42)

where \( v_{BA} \) is the velocity of B relative to a frame AX'Y'Z' attached to A and of fixed orientation (Fig. 15.35). Since A is fixed in this frame, the motion of the body relative to AX'Y'Z' is the motion of a body with a fixed point. The relative velocity \( v_{BA} \) can therefore be obtained from (15.37) after \( r \) has been replaced by the position vector \( r_{BA} \) of B relative to A. Substituting for \( v_{BA} \) into (15.42), we write

\[ v_B = v_A + \omega \times r_{BA} \]  \hspace{1cm} (15.43)

where \( \omega \) is the angular velocity of the body at the instant considered.

The acceleration of B is obtained by a similar reasoning. We first write

\[ a_B = a_A + a_{BA} \]

and, recalling Eq. (15.38),

\[ a_B = a_A + \alpha \times r_{BA} + \omega \times (\omega \times r_{BA}) \]  \hspace{1cm} (15.44)

where \( \alpha \) is the angular acceleration of the body at the instant considered.

Equations (15.43) and (15.44) show that the most general motion of a rigid body is equivalent, at any given instant, to the sum of a translation, in which all the particles of the body have the same velocity and acceleration as a reference particle A, and of a motion in which particle A is assumed to be fixed.†

It is easily shown, by solving (15.43) and (15.44) for \( v_A \) and \( a_A \), that the motion of the body with respect to a frame attached to B would be characterized by the same vectors \( \omega \) and \( \alpha \) as its motion relative to AX'Y'Z'. The angular velocity and angular acceleration of a rigid body at a given instant are thus independent of the choice of reference point. On the other hand, one should keep in mind that whether the moving frame is attached to A or to B, it should maintain a fixed orientation; that is, it should remain parallel to the fixed reference frame OXYZ throughout the motion of the rigid body. In many problems it will be more convenient to use a moving frame which is allowed to rotate as well as to translate. The use of such moving frames will be discussed in Secs. 15.14 and 15.15.

†It is recalled from Sec. 15.12 that, in general, the vectors \( \omega \) and \( \alpha \) are not collinear, and that the accelerations of the particles of the body in their motion relative to the frame AX'Y'Z' cannot be determined as if the body were rotating permanently about the instantaneous axis through A.
SAMPLE PROBLEM 15.11

The crane shown rotates with a constant angular velocity $\omega_1$ of 0.30 rad/s. Simultaneously, the boom is being raised with a constant angular velocity $\omega_2$ of 0.50 rad/s relative to the cab. Knowing that the length of the boom OP is $l = 12$ m, determine (a) the angular velocity $\omega$ of the boom, (b) the angular acceleration $\alpha$ of the boom, (c) the velocity $v$ of the tip of the boom, (d) the acceleration $a$ of the tip of the boom.

SOLUTION

a. Angular Velocity of Boom. Adding the angular velocity $\omega_1$ of the cab and the angular velocity $\omega_2$ of the boom relative to the cab, we obtain the angular velocity $\omega$ of the boom at the instant considered:

$$\omega = \omega_1 + \omega_2 = (0.30 \text{ rad/s})\mathbf{j} + (0.50 \text{ rad/s})\mathbf{k}$$

b. Angular Acceleration of Boom. The angular acceleration $\alpha$ of the boom is obtained by differentiating $\omega$. Since the vector $\omega_1$ is constant in magnitude and direction, we have

$$\alpha = \dot{\omega} = \dot{\omega}_1 + \dot{\omega}_2 = 0 + \dot{\omega}_2$$

where the rate of change $\dot{\omega}_2$ is to be computed with respect to the fixed frame $OXYZ$. However, it is more convenient to use a frame $Oxyz$ attached to the cab and rotating with it, since the vector $\omega_2$ also rotates with the cab and therefore has zero rate of change with respect to that frame. Using Eq. (15.31) with $Q = \omega_2$ and $\Omega = \omega_1$, we write

$$\begin{align*}
\dot{Q}_{OXYZ} &= (\dot{Q})_{Oxyz} + \Omega \times Q \\
\dot{\omega}_2 &= (\dot{\omega}_2)_{Oxyz} + \omega_1 \times \omega_2 \\
\alpha &= (\dot{\omega}_2)_{Oxyz} = 0 + (0.30 \text{ rad/s})\mathbf{j} \times (0.50 \text{ rad/s})\mathbf{k} \\
\alpha &= (0.15 \text{ rad/s}^2)\mathbf{i}
\end{align*}$$

c. Velocity of Tip of Boom. Noting that the position vector of point $P$ is $r = (10.39 \text{ m})\mathbf{i} + (6 \text{ m})\mathbf{j}$ and using the expression found for $\omega$ in part a, we write

$$\begin{align*}
v &= \omega \times r = \\
&= \begin{bmatrix} i & j & k \\ 0 & 0.30 \text{ rad/s} & 0.50 \text{ rad/s} \\ 10.39 \text{ m} & 6 \text{ m} & 0 \end{bmatrix} \\
v &= -(3 \text{ m/s})\mathbf{i} + (5.20 \text{ m/s})\mathbf{j} - (3.21 \text{ m/s})\mathbf{k}
\end{align*}$$

d. Acceleration of Tip of Boom. Recalling that $v = \omega \times r$, we write

$$\begin{align*}
a &= \alpha \times r + \omega \times (\omega \times r) = \alpha \times r + \omega \times v \\
a &= \begin{bmatrix} i & j & k \\ 0.15 & 0 & 0 \\ 10.39 & 6 & 0 \end{bmatrix} \times \begin{bmatrix} i & j & k \\ 0 & 0.30 & 0.50 \\ -3 & 5.20 & -3.12 \end{bmatrix} \\
a &= (0.90 \text{ m/s}^2)\mathbf{k} - (0.94)\mathbf{i} - (2.60)\mathbf{i} - (1.50)\mathbf{j} + (0.90)\mathbf{k} \\
a &= -(3.54 \text{ m/s}^2)\mathbf{i} + (1.50 \text{ m/s}^2)\mathbf{j} + (1.80 \text{ m/s}^2)\mathbf{k}
\end{align*}$$
SAMPLE PROBLEM 15.12

The rod AB, of length 7 in., is attached to the disk by a ball-and-socket connection and to the collar B by a clevis. The disk rotates in the yz plane at a constant rate $\omega_1 = 12 \text{ rad/s}$, while the collar is free to slide along the horizontal rod CD. For the position $\theta = 0$, determine (a) the velocity of the collar, (b) the angular velocity of the rod.

SOLUTION

a. Velocity of Collar. Since point A is attached to the disk and since collar B moves in a direction parallel to the x axis, we have

$$v_A = \omega_1 \times r_A = 12 \mathbf{i} \times 2 \mathbf{k} = -24 \mathbf{j}$$

$$v_B = v_B \mathbf{i}$$

Denoting by $\mathbf{\omega}$ the angular velocity of the rod, we write

$$v_B = v_A + v_{BA} = v_A + \mathbf{\omega} \times r_{BA}$$

$$v_B \mathbf{i} = -24 \mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 6 & 3 & -2 \end{vmatrix}$$

Equating the coefficients of the unit vectors, we obtain

$$v_B = -2 \omega_x - 3 \omega_z$$ (1)

$$24 = 2 \omega_x + 6 \omega_z$$ (2)

$$0 = 3 \omega_x - 6 \omega_z$$ (3)

Multiplying Eqs. (1), (2), (3), respectively, by 6, 3, -2 and adding, we write

$$6v_B + 72 = 0 \quad v_B = 12 \text{ in./s}$$

b. Angular Velocity of Rod AB. We note that the angular velocity cannot be determined from Eqs. (1), (2), and (3), since the determinant formed by the coefficients of $\omega_x$, $\omega_y$, and $\omega_z$ is zero. We must therefore obtain an additional equation by considering the constraint imposed by the clevis at B.

The collar-clevis connection at B permits rotation of AB about the rod CD and also about an axis perpendicular to the plane containing AB and CD. It prevents rotation of AB about the axis EB, which is perpendicular to CD and lies in the plane containing AB and CD. Thus the projection of $\mathbf{\omega}$ on $r_{EB}$ must be zero and we write†

$$\mathbf{\omega} \cdot r_{EB} = 0 \quad (\omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}) \cdot (-3 \mathbf{j} + 2 \mathbf{k}) = 0$$

$$-3 \omega_y + 2 \omega_z = 0$$ (4)

Solving Eqs. (1) through (4) simultaneously, we obtain

$$v_B = -12 \quad \omega_x = 3.69 \quad \omega_y = 1.846 \quad \omega_z = 2.77$$

$$\mathbf{\omega} = (3.69 \text{ rad/s}) \mathbf{i} + (1.846 \text{ rad/s}) \mathbf{j} + (2.77 \text{ rad/s}) \mathbf{k}$$

†We could also note that the direction of EB is that of the vector triple product $r_{EC} \times (r_{EC} \times r_{BA})$ and write $\mathbf{\omega} \cdot [r_{EC} \times (r_{EC} \times r_{BA})] = 0$. This formulation would be particularly useful if the rod CD were skew.
In this lesson you started the study of the kinematics of rigid bodies in three dimensions. You first studied the motion of a rigid body about a fixed point and then the general motion of a rigid body.

A. Motion of a rigid body about a fixed point. To analyze the motion of a point B of a body rotating about a fixed point O you may have to take some or all of the following steps.

1. Determine the position vector r connecting the fixed point O to point B.

2. Determine the angular velocity ω of the body with respect to a fixed frame of reference. The angular velocity ω will often be obtained by adding two component angular velocities ω₁ and ω₂ [Sample Prob. 15.11].

3. Compute the velocity of B by using the equation

   \[ \mathbf{v} = \mathbf{\omega} \times \mathbf{r} \]  

   Your computation will usually be facilitated if you express the vector product as a determinant.

4. Determine the angular acceleration α of the body. The angular acceleration α represents the rate of change \( (\dot{\mathbf{\omega}})_{OXYZ} \) of the vector \( \mathbf{\omega} \) with respect to a fixed frame of reference OXYZ and reflects both a change in magnitude and a change in direction of the angular velocity. However, when computing α you may find it convenient to first compute the rate of change \( (\dot{\mathbf{\omega}})_{Oxyz} \) of \( \mathbf{\omega} \) with respect to a rotating frame of reference Oxyz of your choice and use Eq. (15.31) of the preceding lesson to obtain \( \alpha \). You will write

   \[ \alpha = (\dot{\mathbf{\omega}})_{OXYZ} = (\dot{\mathbf{\omega}})_{Oxyz} + \mathbf{\Omega} \times \mathbf{\omega} \]

   where \( \mathbf{\Omega} \) is the angular velocity of the rotating frame Oxyz [Sample Prob. 15.11].

5. Compute the acceleration of B by using the equation

   \[ \mathbf{a} = \alpha \times \mathbf{r} + \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) \]  

   Note that the vector product \( (\mathbf{\omega} \times \mathbf{r}) \) represents the velocity of point B and was computed in step 3. Also, the computation of the first vector product in (15.38) will be facilitated if you express this product in determinant form. Remember that, as was the case with the plane motion of a rigid body, the instantaneous axis of rotation cannot be used to determine accelerations.
B. General motion of a rigid body. The general motion of a rigid body may be considered as the sum of a translation and a rotation. Keep the following in mind:

a. In the translation part of the motion, all the points of the body have the same velocity \( v_A \) and the same acceleration \( a_A \) as the point \( A \) of the body that has been selected as the reference point.

b. In the rotation part of the motion, the same reference point \( A \) is assumed to be a fixed point.

1. To determine the velocity of a point \( B \) of the rigid body when you know the velocity \( v_A \) of the reference point \( A \) and the angular velocity \( \omega \) of the body, you simply add \( v_A \) to the velocity \( v_{BA} = \omega \times r_{BA} \) of \( B \) in its rotation about \( A \):

\[
v_B = v_A + \omega \times r_{BA}
\]

(15.43)

As indicated earlier, the computation of the vector product will usually be facilitated if you express this product in determinant form.

Equation (15.43) can also be used to determine the magnitude of \( v_B \) when its direction is known, even if \( \omega \) is not known. While the corresponding three scalar equations are linearly dependent and the components of \( \omega \) are indeterminate, these components can be eliminated and \( v_A \) can be found by using an appropriate linear combination of the three equations [Sample Prob. 15.12, part a]. Alternatively, you can assign an arbitrary value to one of the components of \( \omega \) and solve the equations for \( v_A \). However, an additional equation must be sought in order to determine the true values of the components of \( \omega \) [Sample Prob. 15.12, part b].

2. To determine the acceleration of a point \( B \) of the rigid body when you know the acceleration \( a_A \) of the reference point \( A \) and the angular acceleration \( \alpha \) of the body, you simply add \( a_A \) to the acceleration of \( B \) in its rotation about \( A \), as expressed by Eq. (15.38):

\[
a_B = a_A + \alpha \times r_{BA} + \omega \times (\omega \times r_{BA})
\]

(15.44)

Note that the vector product \( (\omega \times r_{BA}) \) represents the velocity \( v_{BA} \) of \( B \) relative to \( A \) and may already have been computed as part of your calculation of \( v_B \). We also remind you that the computation of the other two vector products will be facilitated if you express these products in determinant form.

The three scalar equations associated with Eq. (15.44) can also be used to determine the magnitude of \( a_B \) when its direction is known, even if \( \omega \) and \( \alpha \) are not known. While the components of \( \omega \) and \( \alpha \) are indeterminate, you can assign arbitrary values to one of the components of \( \omega \) and to one of the components of \( \alpha \) and solve the equations for \( a_B \).
15.184 Plate $ABD$ and rod $OB$ are rigidly connected and rotate about the ball-and-socket joint $O$ with an angular velocity $\omega = \omega_i + \omega_j + \omega_k$. Knowing that $v_A = 3 \text{ in./s} \, i + 14 \text{ in./s} \, j + \omega_z \text{ k}$ and $\omega_z = 1.5 \text{ rad/s}$, determine $(a)$ the angular velocity of the assembly, $(b)$ the velocity of point $D$.

![Diagram](image1)

**Fig. P15.184**

15.185 Solve Prob. 15.184, assuming that $\omega_z = -1.5 \text{ rad/s}$.

15.186 At the instant considered the radar antenna shown rotates about the origin of coordinates with an angular velocity $\omega = \omega_i + \omega_j + \omega_z \text{ k}$. Knowing that $(v_A)_y = 300 \text{ mm/s}$, $(v_B)_y = 180 \text{ mm/s}$, and $(v_B)_z = 360 \text{ mm/s}$, determine $(a)$ the angular velocity of the antenna, $(b)$ the velocity of point $A$.

![Diagram](image2)

**Fig. P15.186 and P15.187**

15.187 At the instant considered the radar antenna shown rotates about the origin of coordinates with an angular velocity $\omega = \omega_i + \omega_j + \omega_z \text{ k}$. Knowing that $(v_A)_x = 100 \text{ mm/s}$, $(v_A)_y = -90 \text{ mm/s}$, and $(v_B)_z = 120 \text{ mm/s}$, determine $(a)$ the angular velocity of the antenna, $(b)$ the velocity of point $A$. 
15.188 The blade assembly of an oscillating fan rotates with a constant angular velocity \( \omega_1 = -(360 \text{ rpm})\hat{i} \) with respect to the motor housing. Determine the angular acceleration of the blade assembly, knowing that at the instant shown the angular velocity and angular acceleration of the motor housing are, respectively, \( \omega_2 = -(2.5 \text{ rpm})\hat{j} \) and \( \alpha_2 = 0 \).

![Fig. P15.188](image)

15.189 The rotor of an electric motor rotates at the constant rate \( \omega_1 = 1800 \text{ rpm} \). Determine the angular acceleration of the rotor as the motor is rotated about the \( y \) axis with a constant angular velocity \( \omega_2 \) of 6 rpm counterclockwise when viewed from the positive \( y \) axis.

![Fig. P15.189](image)

15.190 In the system shown, disk \( A \) is free to rotate about the horizontal rod \( OA \). Assuming that shaft \( OC \) is stationary \( (\omega_2 = 0) \), and that shaft \( OC \) rotates with a constant angular velocity \( \omega_1 \), determine (a) the angular velocity of disk \( A \), (b) the angular acceleration of disk \( A \).

![Fig. P15.190 and P15.191](image)

15.191 In the system shown, disk \( A \) is free to rotate about the horizontal rod \( OA \). Assuming that shaft \( OC \) and disk \( B \) rotate with constant angular velocities \( \omega_1 \) and \( \omega_2 \), respectively, both counterclockwise, determine (a) the angular velocity of disk \( A \), (b) the angular acceleration of disk \( A \).

15.192 The L-shaped arm \( BCD \) rotates about the \( z \) axis with a constant angular velocity \( \omega_1 \) of 5 rad/s. Knowing that the 150-mm-radius disk rotates about \( BC \) with a constant angular velocity \( \omega_2 \) of 4 rad/s, determine the angular acceleration of the disk.

![Fig. P15.192](image)

15.193 In Prob. 15.192, determine (a) the velocity of point \( A \), (b) the acceleration of point \( A \).
15.194 A 3-in.-radius disk spins at the constant rate $\omega_2 = 4 \text{ rad/s}$ about an axis held by a housing attached to a horizontal rod that rotates at the constant rate $\omega_1 = 5 \text{ rad/s}$. For the position shown, determine (a) the angular acceleration of the disk, (b) the acceleration of point $P$ on the rim of the disk if $\theta = 0$, (c) the acceleration of point $P$ on the rim of the disk if $\theta = 90^\circ$.

15.195 A 3-in.-radius disk spins at the constant rate $\omega_2 = 4 \text{ rad/s}$ about an axis held by a housing attached to a horizontal rod that rotates at the constant rate $\omega_1 = 5 \text{ rad/s}$. Knowing that $\theta = 30^\circ$, determine the acceleration of point $P$ on the rim of the disk.

15.196 A gun barrel of length $OP = 4 \text{ m}$ is mounted on a turret as shown. To keep the gun aimed at a moving target the azimuth angle $\beta$ is being increased at the rate $d\beta/dt = 30^\circ/\text{s}$ and the elevation angle $\gamma$ is being increased at the rate $d\gamma/dt = 10^\circ/\text{s}$. For the position $\beta = 90^\circ$ and $\gamma = 30^\circ$, determine (a) the angular velocity of the barrel, (b) the angular acceleration of the barrel, (c) the velocity and acceleration of point $P$.

15.197 In the planetary gear system shown, gears $A$ and $B$ are rigidly connected to each other and rotate as a unit about the inclined shaft. Gears $C$ and $D$ rotate with constant angular velocities of $30 \text{ rad/s}$ and $20 \text{ rad/s}$, respectively (both counterclockwise when viewed from the right). Choosing the $x$ axis to the right, the $y$ axis upward, and the $z$ axis pointing out of the plane of the figure, determine (a) the common angular velocity of gears $A$ and $B$, (b) the angular velocity of shaft $FH$, which is rigidly attached to the inclined shaft.
15.198 A 30-mm-radius wheel is mounted on an axle $OB$ of length 100 mm. The wheel rolls without sliding on the horizontal floor, and the axle is perpendicular to the plane of the wheel. Knowing that the system rotates about the $y$ axis at a constant rate $\omega_1 = 2.4$ rad/s, determine (a) the angular velocity of the wheel, (b) the angular acceleration of the wheel, (c) the acceleration of point $C$ located at the highest point on the rim of the wheel.

15.199 Several rods are brazed together to form the robotic guide arm shown which is attached to a ball-and-socket joint at $O$. Rod $OA$ slides in a straight inclined slot while rod $OB$ slides in a slot parallel to the $z$-axis. Knowing that at the instant shown $v_B = (180 \text{ mm/s})\hat{k}$, determine (a) the angular velocity of the guide arm, (b) the velocity of point $A$, (c) the velocity of point $C$.

15.200 In Prob. 15.199 the speed of point $B$ is known to be constant. For the position shown, determine (a) the angular acceleration of the guide arm, (b) the acceleration of point $C$.

15.201 The 45° sector of a 10-in.-radius circular plate is attached to a fixed ball-and-socket joint at $O$. As edge $OA$ moves on the horizontal surface, edge $OB$ moves along the vertical wall. Knowing that point $A$ moves at a constant speed of 60 in./s, determine for the position shown (a) the angular velocity of the plate, (b) the velocity of point $B$.

15.202 Rod $AB$ of length 275 mm is connected by ball-and-socket joints to collars $A$ and $B$, which slide along the two rods shown. Knowing that collar $B$ moves toward the origin $O$ at a constant speed of 180 mm/s, determine the velocity of collar $A$ when $c = 175$ mm.

15.203 Rod $AB$ of length 275 mm is connected by ball-and-socket joints to collars $A$ and $B$, which slide along the two rods shown. Knowing that collar $B$ moves toward the origin $O$ at a constant speed of 180 mm/s, determine the velocity of collar $A$ when $c = 50$ mm.
15.204 Rod $AB$ is connected by ball-and-socket joints to collar $A$ and to the 16-in.-diameter disk $C$. Knowing that disk $C$ rotates counterclockwise at the constant rate $\omega_0 = 3$ rad/s in the $zx$ plane, determine the velocity of collar $A$ for the position shown.

15.205 Rod $AB$ of length 29 in. is connected by ball-and-socket joints to the rotating crank $BC$ and to the collar $A$. Crank $BC$ is of length 8 in. and rotates in the horizontal $xy$ plane at the constant rate $\omega_0 = 10$ rad/s. At the instant shown, when crank $BC$ is parallel to the $z$ axis, determine the velocity of collar $A$.

15.206 Rod $AB$ of length 300 mm is connected by ball-and-socket joints to collars $A$ and $B$, which slide along the two rods shown. Knowing that collar $B$ moves toward point $D$ at a constant speed of 50 mm/s, determine the velocity of collar $A$ when $c = 80$ mm.

15.207 Rod $AB$ of length 300 mm is connected by ball-and-socket joints to collars $A$ and $B$, which slide along the two rods shown. Knowing that collar $B$ moves toward point $D$ at a constant speed of 50 mm/s, determine the velocity of collar $A$ when $c = 120$ mm.
15.208 Rod AB of length 25 in. is connected by ball-and-socket joints to collars A and B, which slide along the two rods shown. Knowing that collar B moves toward point E at a constant speed of 20 in./s, determine the velocity of collar A as collar B passes through point D.

15.209 Rod AB of length 25 in. is connected by ball-and-socket joints to collars A and B, which slide along the two rods shown. Knowing that collar B moves toward point E at a constant speed of 20 in./s, determine the velocity of collar A as collar B passed through point C.

15.210 Two shafts AC and EG, which lie in the vertical \( yz \) plane, are connected by a universal joint at D. Shaft AC rotates with a constant angular velocity \( \omega_1 \) as shown. At a time when the arm of the crosspiece attached to shaft AC is vertical, determine the angular velocity of shaft EG.

15.211 Solve Prob. 15.210, assuming that the arm of the crosspiece attached to the shaft AC is horizontal.

15.212 In Prob. 15.203, the ball-and-socket joint between the rod and collar A is replaced by the clevis shown. Determine (a) the angular velocity of the rod, (b) the velocity of collar A.

15.213 In Prob. 15.204, the ball-and-socket joint between the rod and collar A is replaced by the clevis shown. Determine (a) the angular velocity of the rod, (b) the velocity of collar A.
15.214 through 15.219  For the mechanism of the problem indicated, determine the acceleration of collar A.


15.215  Mechanism of Prob. 15.203.

15.216  Mechanism of Prob. 15.204.

15.217  Mechanism of Prob. 15.205.

15.218  Mechanism of Prob. 15.206.

15.219  Mechanism of Prob. 15.207.

*15.14  THREE-DIMENSIONAL MOTION OF A PARTICLE RELATIVE TO A ROTATING FRAME. CORIOLIS ACCELERATION

We saw in Sec. 15.10 that given a vector function \( Q(t) \) and two frames of reference centered at \( O \)—a fixed frame \( OXYZ \) and a rotating frame \( Oxyz \)—the rates of change of \( Q \) with respect to the two frames satisfy the relation

\[
(\dot{Q})_{OXYZ} = (\dot{Q})_{Oxyz} + \Omega \times Q
\]

We had assumed at the time that the frame \( Oxyz \) was constrained to rotate about a fixed axis \( OA \). However, the derivation given in Sec. 15.10 remains valid when the frame \( Oxyz \) is constrained only to have a fixed point \( O \). Under this more general assumption, the axis \( OA \) represents the instantaneous axis of rotation of the frame \( Oxyz \) (Sec. 15.12) and the vector \( \Omega \), its angular velocity at the instant considered (Fig. 15.36).

Let us now consider the three-dimensional motion of a particle \( P \) relative to a rotating frame \( Oxyz \) constrained to have a fixed origin \( O \). Let \( r \) be the position vector of \( P \) at a given instant and \( \Omega \) be the angular velocity of the frame \( Oxyz \) with respect to the fixed frame \( OXYZ \) at the same instant (Fig. 15.37). The derivations given in Sec. 15.11 for the two-dimensional motion of a particle can be readily extended to the three-dimensional case, and the absolute velocity \( v_P \) of \( P \) (i.e., its velocity with respect to the fixed frame \( OXYZ \)) can be expressed as

\[
v_P = \Omega \times r + (\dot{r})_{Oxyz}
\]

Denoting by \( \mathcal{F} \) the rotating frame \( Oxyz \), we write this relation in the alternative form

\[
v_P = v_P' + v_{P/\mathcal{F}}
\]

where \( v_P' \) = absolute velocity of particle \( P \)
\( v_P' \) = velocity of point \( P' \) of moving frame \( \mathcal{F} \) coinciding with \( P \)
\( v_{P/\mathcal{F}} \) = velocity of \( P \) relative to moving frame \( \mathcal{F} \)

The absolute acceleration \( a_P \) of \( P \) can be expressed as

\[
a_P = \dot{\Omega} \times r + \Omega \times (\Omega \times r) + 2\Omega \times (\dot{r})_{Oxyz} + (\ddot{r})_{Oxyz}
\]
An alternative form is

\[ \mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/P} + \mathbf{a}_c \]  

where \( \mathbf{a}_P \) = absolute acceleration of particle \( P \)
\( \mathbf{a}_{P'} \) = acceleration of point \( P' \) of moving frame \( \mathcal{F} \) coinciding with \( P \)
\( \mathbf{a}_{P/P} \) = acceleration of \( P \) relative to moving frame \( \mathcal{F} \)

\[ \mathbf{a}_c = 2\mathbf{\Omega} \times (\mathbf{\dot{r}})_{Oxyz} = 2\mathbf{\Omega} \times \mathbf{v}_{P/P} = \text{complementary, or Coriolis, acceleration} \]

We note that the Coriolis acceleration is perpendicular to the vectors \( \mathbf{\Omega} \) and \( \mathbf{v}_{P/P} \). However, since these vectors are usually not perpendicular to each other, the magnitude of \( \mathbf{a}_c \) is in general not equal to \( 2\mathbf{\Omega}\mathbf{v}_{P/P} \), as was the case for the plane motion of a particle. We further note that the Coriolis acceleration reduces to zero when the vectors \( \mathbf{\Omega} \) and \( \mathbf{v}_{P/P} \) are parallel, or when either of them is zero.

Rotating frames of reference are particularly useful in the study of the three-dimensional motion of rigid bodies. If a rigid body has a fixed point \( O \), as was the case for the crane of Sample Prob. 15.11, we can use a frame \( Oxyz \) which is neither fixed nor rigidly attached to the rigid body. Denoting by \( \mathbf{\Omega} \) the angular velocity of the frame \( Oxyz \), we then resolve the angular velocity \( \mathbf{\omega} \) of the body into the components \( \mathbf{\Omega} \) and \( \mathbf{\omega}_{P/P} \), where the second component represents the angular velocity of the body relative to the frame \( Oxyz \) (see Sample Prob. 15.14). An appropriate choice of the rotating frame often leads to a simpler analysis of the motion of the rigid body than would be possible with axes of fixed orientation. This is especially true in the case of the general three-dimensional motion of a rigid body, i.e., when the rigid body under consideration has no fixed point (see Sample Prob. 15.15).

### 15.15 Frame of Reference in General Motion

Consider a fixed frame of reference \( OXYZ \) and a frame \( Axyz \) which moves in a known, but arbitrary, fashion with respect to \( OXYZ \) (Fig. 15.38). Let \( P \) be a particle moving in space. The position of \( P \) is defined at any instant by the vector \( \mathbf{r}_P \) in the fixed frame, and by the vector \( \mathbf{r}_{P/A} \) in the moving frame. Denoting by \( \mathbf{r}_A \) the position vector of \( A \) in the fixed frame, we have

\[ \mathbf{r}_P = \mathbf{r}_A + \mathbf{r}_{P/A} \]  

(15.49)

The absolute velocity \( \mathbf{v}_P \) of the particle is obtained by writing

\[ \mathbf{v}_P = \mathbf{\dot{r}}_P = \mathbf{\dot{r}}_A + \mathbf{\dot{r}}_{P/A} \]  

(15.50)

where the derivatives are defined with respect to the fixed frame \( OXYZ \). The first term in the right-hand member of (15.50) thus represents the velocity \( \mathbf{v}_A \) of the origin \( A \) of the moving axes. On the other hand, since the rate of change of a vector is the same with

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†It is important to note the difference between Eq. (15.48) and Eq. (15.21) of Sec. 15.8. See the footnote on page 974.
respect to a fixed frame and with respect to a frame in translation (Sec. 11.10), the second term can be regarded as the velocity \( \mathbf{v}_{PA} \) of \( P \) relative to the frame \( AX'Y'Z' \) of the same orientation as \( OXYZ \) and the same origin as \( Axyz \). We therefore have

\[
\mathbf{v}_P = \mathbf{v}_A + \mathbf{v}_{PA} \quad (15.51)
\]

But the velocity \( \mathbf{v}_{PA} \) of \( P \) relative to \( AX'Y'Z' \) can be obtained from (15.45) by substituting \( \mathbf{r}_{PA} \) for \( \mathbf{r} \) in that equation. We write

\[
\mathbf{v}_P = \mathbf{v}_A + \mathbf{\Omega} \times \mathbf{r}_{PA} + (\dot{\mathbf{r}}_{PA})_{Axyz} \quad (15.52)
\]

where \( \mathbf{\Omega} \) is the angular velocity of the frame \( Axyz \) at the instant considered.

The absolute acceleration \( \mathbf{a}_P \) of the particle is obtained by differentiating (15.51) and writing

\[
\mathbf{a}_P = \ddot{\mathbf{v}}_P = \ddot{\mathbf{v}}_A + \ddot{\mathbf{v}}_{PA} \quad (15.53)
\]

where the derivatives are defined with respect to either of the frames \( OXYZ \) or \( AX'Y'Z' \). Thus, the first term in the right-hand member of (15.53) represents the acceleration \( \mathbf{a}_A \) of the origin \( A \) of the moving axes and the second term represents the acceleration \( \mathbf{a}_{PA} \) of \( P \) relative to the frame \( AX'Y'Z' \). This acceleration can be obtained from (15.47) by substituting \( \mathbf{r}_{PA} \) for \( \mathbf{r} \). We therefore write

\[
\mathbf{a}_P = \mathbf{a}_A + \dot{\mathbf{\Omega}} \times \mathbf{r}_{PA} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{PA}) + 2\mathbf{\Omega} \times (\dot{\mathbf{r}}_{PA})_{Axyz} + (\ddot{\mathbf{r}}_{PA})_{Axyz} \quad (15.54)
\]

Formulas (15.52) and (15.54) make it possible to determine the velocity and acceleration of a given particle with respect to a fixed frame of reference, when the motion of the particle is known with respect to a moving frame. These formulas become more significant, and considerably easier to remember, if we note that the sum of the first two terms in (15.52) represents the velocity of the point \( P' \) of the moving frame which coincides with \( P \) at the instant considered, and that the sum of the first three terms in (15.54) represents the acceleration of the same point. Thus, the relations (15.46) and (15.48) of the preceding section are still valid in the case of a reference frame in general motion, and we can write

\[
\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P'\beta} \quad (15.46)
\]

\[
\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P'\beta} + \mathbf{a}_c \quad (15.48)
\]

where the various vectors involved have been defined in Sec. 15.14.

It should be noted that if the moving reference frame \( \mathcal{F} \) (or \( Axyz \)) is in translation, the velocity and acceleration of the point \( P' \) of the frame which coincides with \( P \) become, respectively, equal to the velocity and acceleration of the origin \( A \) of the frame. On the other hand, since the frame maintains a fixed orientation, \( \mathbf{a}_c \) is zero, and the relations (15.46) and (15.48) reduce, respectively, to the relations (11.33) and (11.34) derived in Sec. 11.12.
SAMPLE PROBLEM 15.13

The bent rod $OAB$ rotates about the vertical $OB$. At the instant considered, its angular velocity and angular acceleration are, respectively, $20 \text{ rad/s}$ and $200 \text{ rad/s}^2$, both clockwise when viewed from the positive $Y$ axis. The collar $D$ moves along the rod, and at the instant considered, $OD = 8$ in. The velocity and acceleration of the collar relative to the rod are, respectively, $50 \text{ in./s}$ and $600 \text{ in./s}^2$, both upward. Determine (a) the velocity of the collar, (b) the acceleration of the collar.

**SOLUTION**

Frames of Reference. The frame $OXYZ$ is fixed. We attach the rotating frame $Oxyz$ to the bent rod. Its angular velocity and angular acceleration relative to $OXYZ$ are therefore $\Omega = (-20 \text{ rad/s}) \hat{j}$ and $\dot{\Omega} = (-200 \text{ rad/s}^2) \hat{j}$, respectively. The position vector of $D$ is

$$
\mathbf{r} = (8 \text{ in.})(\sin 30^\circ \mathbf{i} + \cos 30^\circ \mathbf{j}) = (4 \text{ in.})\mathbf{i} + (6.93 \text{ in.})\mathbf{j}
$$

**a. Velocity $\mathbf{v}_D$.** Denoting by $D'$ the point of the rod which coincides with $D$ and by $\mathbf{F}$ the rotating frame $Oxyz$, we write from Eq. (15.46)

$$
\mathbf{v}_D = \mathbf{v}_{D'} + \mathbf{v}_{D/F}
$$

where

$$
\mathbf{v}_{D'} = \Omega \times \mathbf{r} = (-20 \text{ rad/s})\hat{j} \times [(4 \text{ in.})\mathbf{i} + (6.93 \text{ in.})\mathbf{j}] = (80 \text{ in./s})\hat{k}
$$

$$
\mathbf{v}_{D/F} = (50 \text{ in./s})(\sin 30^\circ \mathbf{i} + \cos 30^\circ \mathbf{j}) = (25 \text{ in./s})\mathbf{i} + (43.3 \text{ in./s})\mathbf{j}
$$

Substituting the values obtained for $\mathbf{v}_{D'}$ and $\mathbf{v}_{D/F}$ into (1), we find

$$
\mathbf{v}_D = (25 \text{ in./s})\mathbf{i} + (43.3 \text{ in./s})\mathbf{j} + (80 \text{ in./s})\hat{k}
$$

**b. Acceleration $\mathbf{a}_D$.** From Eq. (15.48) we write

$$
\mathbf{a}_D = \mathbf{a}_{D'} + \mathbf{a}_{D/F} + \mathbf{a}_c
$$

where

$$
\mathbf{a}_{D'} = \dot{\Omega} \times \mathbf{r} + \Omega \times (\dot{\Omega} \times \mathbf{r})
= (-200 \text{ rad/s}^2)\hat{j} \times [(4 \text{ in.})\mathbf{i} + (6.93 \text{ in.})\mathbf{j}] - (20 \text{ rad/s})\hat{j} \times (80 \text{ in./s})\hat{k}
= + (800 \text{ in./s}^2)\mathbf{i} - (1600 \text{ in./s}^2)\mathbf{j}
$$

$$
\mathbf{a}_{D/F} = (600 \text{ in./s}^2)(\sin 30^\circ \mathbf{i} + \cos 30^\circ \mathbf{j}) = (300 \text{ in./s}^2)\mathbf{i} + (520 \text{ in./s}^2)\mathbf{j}
$$

$$
\mathbf{a}_c = 2\Omega \times \mathbf{v}_{D/F}
= 2(-20 \text{ rad/s})\hat{j} \times [(25 \text{ in./s})\mathbf{i} + (43.3 \text{ in./s})\mathbf{j}] = (1000 \text{ in./s}^2)\mathbf{k}
$$

Substituting the values obtained for $\mathbf{a}_{D'}$, $\mathbf{a}_{D/F}$, and $\mathbf{a}_c$ into (2),

$$
\mathbf{a}_D = -(1300 \text{ in./s}^2)\mathbf{i} + (520 \text{ in./s}^2)\mathbf{j} + (1800 \text{ in./s}^2)\hat{k}
$$
SAMPLE PROBLEM 15.14

The crane shown rotates with a constant angular velocity \( \omega_1 \) of 0.30 rad/s. Simultaneously, the boom is being raised with a constant angular velocity \( \omega_2 \) of 0.50 rad/s relative to the cab. Knowing that the length of the boom OP is \( l = 12 \text{ m} \), determine (a) the velocity of the tip of the boom, (b) the acceleration of the tip of the boom.

SOLUTION

Frames of Reference. The frame OXYZ is fixed. We attach the rotating frame Ox'yz' to the cab. Its angular velocity with respect to the frame OXYZ is therefore \( \Omega = \omega_1 = (0.30 \text{ rad/s}) \hat{j} \). The angular velocity of the boom relative to the cab and the rotating frame Ox'yz' (or \( \mathbf{F} \), for short) is \( \omega_{B/F} = \omega_2 = (0.50 \text{ rad/s}) \hat{k} \).

a. Velocity \( v_P \). From Eq. (15.46) we write

\[
v_P = v_{P'} + v_{P/F} \tag{1}
\]

where \( v_{P'} \) is the velocity of the point \( P' \) of the rotating frame which coincides with \( P \):

\[
v_{P'} = \Omega \times r = (0.30 \text{ rad/s}) \hat{j} \times [(10.39 \text{ m}) \hat{i} + (6 \text{ m}) \hat{j}] = -(3.12 \text{ m/s}) \hat{k}
\]

and where \( v_{P/F} \) is the velocity of \( P \) relative to the rotating frame Ox'yz'. But the angular velocity of the boom relative to Ox'yz was found to be \( \omega_{B/F} = (0.50 \text{ rad/s}) \hat{k} \). The velocity of its tip \( P \) relative to Ox'yz is therefore

\[
v_{P/F} = \omega_{B/F} \times r = (0.50 \text{ rad/s}) \hat{k} \times [(10.39 \text{ m}) \hat{i} + (6 \text{ m}) \hat{j}]
\]

\[
= -(3 \text{ m/s}) \hat{i} + (5.20 \text{ m/s}) \hat{j}
\]

Substituting the values obtained for \( v_{P'} \) and \( v_{P/F} \) into (1), we find

\[
v_P = -(3 \text{ m/s}) \hat{i} + (5.20 \text{ m/s}) \hat{j} - (3.12 \text{ m/s}) \hat{k}
\]

b. Acceleration \( a_P \). From Eq. (15.48) we write

\[
a_P = a_{P'} + a_{P/F} + a_c \tag{2}
\]

Since \( \Omega \) and \( \omega_{B/F} \) are both constant, we have

\[
a_{P'} = \Omega \times (\Omega \times r) = (0.30 \text{ rad/s}) \hat{j} \times (-3.12 \text{ m/s}) \hat{k} = -(0.94 \text{ m/s}^2) \hat{i}
\]

\[
a_{P/F} = \omega_{B/F} \times (\omega_{B/F} \times r) = (0.50 \text{ rad/s}) \hat{k} \times [-(3 \text{ m/s}) \hat{i} + (5.20 \text{ m/s}) \hat{j}]
\]

\[
= -(1.50 \text{ m/s}^2) \hat{j} - (2.60 \text{ m/s}^2) \hat{i}
\]

\[
a_c = 2\Omega \times v_{P/F} = 2(0.30 \text{ rad/s}) \hat{j} \times [-(3 \text{ m/s}) \hat{i} + (5.20 \text{ m/s}) \hat{j}]
\]

\[
= (1.80 \text{ m/s}^2) \hat{k}
\]

Substituting for \( a_{P'} \), \( a_{P/F} \), and \( a_c \) into (2), we find

\[
a_P = -(3.54 \text{ m/s}^2) \hat{i} - (1.50 \text{ m/s}^2) \hat{j} + (1.80 \text{ m/s}^2) \hat{k}
\]
SAMPLE PROBLEM 15.15

Disk $D$, of radius $R$, is pinned to end $A$ of the arm $OA$ of length $L$ located in the plane of the disk. The arm rotates about a vertical axis through $O$ at the constant rate $\omega_1$, and the disk rotates about $A$ at the constant rate $\omega_2$. Determine (a) the velocity of point $P$ located directly above $A$, (b) the acceleration of $P$, and (c) the angular velocity and angular acceleration of the disk.

SOLUTION

Frames of Reference. The frame $OXYZ$ is fixed. We attach the moving frame $Axyz$ to the arm $OA$. Its angular velocity with respect to the frame $OXYZ$ is therefore $\Omega = \omega_1 \hat{j}$. The angular velocity of disk $D$ relative to the moving frame $Axyz$ (or $\mathcal{F}$, for short) is $\omega_{D|\mathcal{F}} = \omega_2 \hat{k}$. The position vector of $P$ relative to $O$ is $\mathbf{r} = L \hat{i} + R \hat{j}$, and its position vector relative to $A$ is $\mathbf{r}_{PA} = R \hat{j}$.

a. Velocity $\mathbf{v}_p$. Denoting by $P'$ the point of the moving frame which coincides with $P$, we write from Eq. (15.46)$\n\begin{equation}
\mathbf{v}_p = \mathbf{v}_{P'} + \mathbf{v}_{P|\mathcal{F}} \tag{1}
\end{equation}$
where $\mathbf{v}_{P'} = \Omega \times \mathbf{r} = \omega_1 \hat{j} \times (L \hat{i} + R \hat{j}) = -\omega_1 L \hat{k}$
\[\mathbf{v}_{P|\mathcal{F}} = \omega_{D|\mathcal{F}} \times \mathbf{r}_{PA} = \omega_2 \hat{k} \times R \hat{j} = -\omega_2 R \hat{i}\]
Substituting the values obtained for $\mathbf{v}_{P'}$ and $\mathbf{v}_{P|\mathcal{F}}$ into (1), we find
\[\mathbf{v}_p = -\omega_2 R \hat{i} - \omega_1 L \hat{k}\]

b. Acceleration $\mathbf{a}_p$. From Eq. (15.48) we write
\[\mathbf{a}_p = \mathbf{a}_{P'} + \mathbf{a}_{P|\mathcal{F}} + \mathbf{a}_s\]  \(\tag{2}\)
Since $\Omega$ and $\omega_{D|\mathcal{F}}$ are both constant, we have
\[\mathbf{a}_{P'} = \Omega \times (\Omega \times \mathbf{r}) = \omega_1 \hat{j} \times (-\omega_1 \hat{k} \times L \hat{i}) = -\omega_1^2 L \hat{i}\]
\[\mathbf{a}_{P|\mathcal{F}} = \omega_{D|\mathcal{F}} \times (\omega_{D|\mathcal{F}} \times \mathbf{r}_{PA}) = \omega_2 \hat{k} \times (-\omega_2 R \hat{i}) = -\omega_2^2 R \hat{j}\]
\[\mathbf{a}_s = 2\Omega \times \mathbf{v}_{P|\mathcal{F}} = 2\omega_1 \hat{j} \times (-\omega_2 R \hat{i}) = 2\omega_1 \omega_2 R \hat{k}\]
Substituting the values obtained into (2), we find
\[\mathbf{a}_p = -\omega_1^2 L \hat{i} - \omega_2^2 R \hat{j} + 2\omega_1 \omega_2 R \hat{k}\]

c. Angular Velocity and Angular Acceleration of Disk.
\[\omega = \Omega + \omega_{D|\mathcal{F}} \quad \omega = \omega_1 \hat{j} + \omega_2 \hat{k}\]

Using Eq. (15.31) with $Q = \omega$, we write
\[\alpha = (\dot{\omega})_{OXYZ} = (\dot{\omega})_{Axyz} + \Omega \times \omega = 0 + \omega_1 \hat{j} \times (\omega_1 \hat{j} + \omega_2 \hat{k})\]
\[\alpha = \omega_1 \omega_2 \hat{i}\]
SOLVING PROBLEMS ON YOUR OWN

In this lesson you concluded your study of the kinematics of rigid bodies by learning how to use an auxiliary frame of reference $\mathcal{F}$ to analyze the three-dimensional motion of a rigid body. This auxiliary frame may be a rotating frame with a fixed origin $O$, or it may be a frame in general motion.

A. Using a rotating frame of reference. As you approach a problem involving the use of a rotating frame $\mathcal{F}$ you should take the following steps.

1. Select the rotating frame $\mathcal{F}$ that you wish to use and draw the corresponding coordinate axes $x$, $y$, and $z$ from the fixed point $O$.

2. Determine the angular velocity $\Omega$ of the frame $\mathcal{F}$ with respect to a fixed frame $OXYZ$. In most cases, you will have selected a frame which is attached to some rotating element of the system; $\Omega$ will then be the angular velocity of that element.

3. Designate as $P'$ the point of the rotating frame $\mathcal{F}$ that coincides with the point $P$ of interest at the instant you are considering. Determine the velocity $v_{P'}$ and the acceleration $a_{P'}$ of point $P'$. Since $P'$ is part of $\mathcal{F}$ and has the same position vector $\mathbf{r}$ as $P$, you will find that

$$v_{P'} = \Omega \times \mathbf{r} \quad \text{and} \quad a_{P'} = \alpha \times \mathbf{r} + \Omega \times (\Omega \times \mathbf{r})$$

where $\alpha$ is the angular acceleration of $\mathcal{F}$. However, in many of the problems that you will encounter, the angular velocity of $\mathcal{F}$ is constant in both magnitude and direction, and $\alpha = 0$.

4. Determine the velocity and acceleration of point $P$ with respect to the frame $\mathcal{F}$. As you are trying to determine $v_{P\mathcal{B}}$ and $a_{P\mathcal{B}}$ you will find it useful to visualize the motion of $P$ on frame $\mathcal{F}$ when the frame is not rotating. If $P$ is a point of a rigid body $\mathcal{B}$ which has an angular velocity $\omega_{\mathcal{B}}$ and an angular acceleration $\alpha_{\mathcal{B}}$ relative to $\mathcal{F}$ [Sample Prob. 15.14], you will find that

$$v_{P\mathcal{B}} = \omega_{\mathcal{B}} \times \mathbf{r} \quad \text{and} \quad a_{P\mathcal{B}} = \alpha_{\mathcal{B}} \times \mathbf{r} + \omega_{\mathcal{B}} \times (\omega_{\mathcal{B}} \times \mathbf{r})$$

In many of the problems that you will encounter, the angular velocity of body $\mathcal{B}$ relative to frame $\mathcal{F}$ is constant in both magnitude and direction, and $\alpha_{\mathcal{B}} = 0$.

5. Determine the Coriolis acceleration. Considering the angular velocity $\Omega$ of frame $\mathcal{F}$ and the velocity $v_{P\mathcal{F}}$ of point $P$ relative to that frame, which was computed in the previous step, you write

$$a_c = 2\Omega \times v_{P\mathcal{F}}$$
6. The velocity and the acceleration of P with respect to the fixed frame OXYZ can now be obtained by adding the expressions you have determined:

\[
\begin{align*}
\mathbf{v}_P &= \mathbf{v}_P + \mathbf{v}_{P/A} \\
\mathbf{a}_P &= \mathbf{a}_P + \mathbf{a}_{P/A} + \mathbf{a}_r
\end{align*}
\]  
(15.46)  
(15.48)

B. Using a frame of reference in general motion. The steps that you will take differ only slightly from those listed under A. They consist of the following:

1. Select the frame \( \mathcal{F} \) that you wish to use and a reference point A in that frame, from which you will draw the coordinate axes, \( x, y, \) and \( z \) defining that frame. You will consider the motion of the frame as the sum of a translation with A and a rotation about A.

2. Determine the velocity \( \mathbf{v}_A \) of point A and the angular velocity \( \Omega \) of the frame. In most cases, you will have selected a frame which is attached to some element of the system; \( \Omega \) will then be the angular velocity of that element.

3. Designate as \( P' \) the point of frame \( \mathcal{F} \) that coincides with the point P of interest at the instant you are considering, and determine the velocity \( \mathbf{v}_P \) and the acceleration \( \mathbf{a}_P \) of that point. In some cases, this can be done by visualizing the motion of P if that point were prevented from moving with respect to \( \mathcal{F} \) [Sample Prob. 15.15]. A more general approach is to recall that the motion of \( P' \) is the sum of a translation with the reference point A and a rotation about A. The velocity \( \mathbf{v}_P \) and the acceleration \( \mathbf{a}_P \) of \( P' \), therefore, can be obtained by adding \( \mathbf{v}_A \) and \( \mathbf{a}_A \), respectively, to the expressions found in paragraph A3 and replacing the position vector \( \mathbf{r} \) by the vector \( \mathbf{r}_{P/A} \) drawn from A to P:

\[
\begin{align*}
\mathbf{v}_P &= \mathbf{v}_A + \Omega \times \mathbf{r}_{P/A} \\
\mathbf{a}_P &= \mathbf{a}_A + \alpha \times \mathbf{r}_{P/A} + \Omega \times (\Omega \times \mathbf{r}_{P/A})
\end{align*}
\]

Steps 4, 5, and 6 are the same as in Part A, except that the vector \( \mathbf{r} \) should again be replaced by \( \mathbf{r}_{P/A} \). Thus, Eqs. (15.46) and (15.48) can still be used to obtain the velocity and the acceleration of P with respect to the fixed frame of reference OXYZ.
15.220 Rod AB is welded to the 12-in.-radius plate which rotates at the constant rate \( \omega_1 = 6 \text{ rad/s} \). Knowing that collar D moves toward end B of the rod at a constant speed \( u = 78 \text{ in./s} \), determine, for the position shown, (a) the velocity of D, (b) the acceleration of D.

![Fig. P15.220](image)

15.221 The bent rod shown rotates at the constant rate \( \omega_1 = 3 \text{ rad/s} \). Knowing that collar C moves toward point D at a constant relative speed \( u = 34 \text{ in./s} \), determine, for the position shown, the velocity and acceleration of C if (a) \( x = 5 \text{ in.} \), (b) \( x = 15 \text{ in.} \).

![Fig. P15.221](image)

15.222 The circular plate shown rotates about its vertical diameter at the constant rate \( \omega_1 = 10 \text{ rad/s} \). Knowing that in the position shown the disk lies in the XY plane and point D of strap CD moves upward at a constant relative speed \( u = 1.5 \text{ m/s} \), determine (a) the velocity of D, (b) the acceleration of D.

![Fig. P15.222](image)
15.223 Solve Prob. 15.222, assuming that, at the instant shown, the angular velocity \( \omega_1 \) of the plate is 10 rad/s and is decreasing at the rate of 25 rad/s\(^2\), while the relative speed \( u \) of point \( D \) of strap \( CD \) is 1.5 m/s and is decreasing at the rate 3 m/s\(^2\).

15.224 A square plate of side 18 in. is hinged at \( A \) and \( B \) to a clevis. The plate rotates at the constant rate \( \omega_2 = 4 \) rad/s with respect to the clevis, which itself rotates at the constant rate \( \omega_1 = 3 \) rad/s about the \( Y \) axis. For the position shown, determine (a) the velocity of point \( C \), (b) the acceleration of point \( C \).

![Fig. P15.224 and P15.225](image)

15.225 A square plate of side 18 in. is hinged at \( A \) and \( B \) to a clevis. The plate rotates at the constant rate \( \omega_2 = 4 \) rad/s with respect to the clevis, which itself rotates at the constant rate \( \omega_1 = 3 \) rad/s about the \( Y \) axis. For the position shown, determine (a) the velocity of corner \( D \), (b) the acceleration of corner \( D \).

15.226 through 15.228 The rectangular plate shown rotates at the constant rate \( \omega_2 = 12 \) rad/s with respect to arm \( AE \), which itself rotates at the constant rate \( \omega_1 = 9 \) rad/s about the \( Z \) axis. For the position shown, determine the velocity and acceleration of the point of the plate indicated.

15.226 Corner \( B \).
15.227 Point \( D \).
15.228 Corner \( C \).

15.229 Solve Prob. 15.228, assuming that at the instant shown the angular velocity \( \omega_2 \) of the plate with respect to arm \( AE \) is 12 rad/s and is decreasing at the rate of 60 rad/s\(^2\), while the angular velocity \( \omega_1 \) of the arm about the \( Z \) axis is 9 rad/s and is decreasing at the rate of 45 rad/s\(^2\).

15.230 Solve Prob. 15.221, assuming that at the instant shown the angular velocity \( \omega_1 \) of the rod is 3 rad/s and is increasing at the rate of 12 rad/s\(^2\), while the relative speed \( u \) of the collar is 34 in./s and is decreasing at the rate of 85 in./s\(^2\).

15.231 Using the method of Sec. 15.14, solve Prob. 15.191.

15.232 Using the method of Sec. 15.14, solve Prob. 15.195.
15.233 Using the method of Sec. 15.14, solve Prob. 15.192.

15.234 The body AB and rod BC of the robotic component shown rotate at the constant rate \( \omega_1 = 0.60 \text{ rad/s} \) about the Y axis. Simultaneously a wire-and-pulley control causes arm CD to rotate about C at the constant rate \( \omega = d\beta/dt = 0.45 \text{ rad/s} \). Knowing \( \beta = 120^\circ \), determine (a) the angular acceleration of arm CD, (b) the velocity of D, (c) the acceleration of D.

![Diagram of robotic component with labeled parts A, B, C, D, and rotation axes](image)

Fig. P15.234

15.235 A disk of radius 120 mm rotates at the constant rate \( \omega_2 = 5 \text{ rad/s} \) with respect to the arm AB, which itself rotates at the constant rate \( \omega_1 = 3 \text{ rad/s} \). For the position shown, determine the velocity and acceleration of point C.

![Diagram of disk and arm with labeled parts A, B, C, D, and rotation axes](image)

Fig. P15.235 and P15.236

15.236 A disk of radius 120 mm rotates at the constant rate \( \omega_2 = 5 \text{ rad/s} \) with respect to the arm AB, which itself rotates at the constant rate \( \omega_1 = 3 \text{ rad/s} \). For the position shown, determine the velocity and acceleration of point D.

15.237 The crane shown rotates at the constant rate \( \omega_1 = 0.25 \text{ rad/s} \); simultaneously, the telescoping boom is being lowered at the constant rate \( \omega_2 = 0.40 \text{ rad/s} \). Knowing that at the instant shown the length of the boom is 20 ft and is increasing at the constant rate \( u = 1.5 \text{ ft/s} \), determine the velocity and acceleration of point B.
15.238 The arm \( AB \) of length 5 m is used to provide an elevated platform for construction workers. In the position shown, arm \( AB \) is being raised at the constant rate \( \frac{d\theta}{dt} = 0.25 \text{ rad/s} \); simultaneously, the unit is being rotated about the \( Y \) axis at the constant rate \( \omega_1 = 0.15 \text{ rad/s} \). Knowing that \( \theta = 20^\circ \), determine the velocity and acceleration of point \( B \).

![Fig. P15.238](image)

15.239 Solve Prob. 15.238, assuming that \( \theta = 40^\circ \).

15.240 A disk of 180-mm radius rotates at the constant rate \( \omega_2 = 12 \text{ rad/s} \) with respect to arm \( CD \), which itself rotates at the constant rate \( \omega_1 = 8 \text{ rad/s} \) about the \( Y \) axis. Determine at the instant shown the velocity and acceleration of point \( A \) on the rim of the disk.

![Fig. P15.240 and P15.241](image)

15.241 A disk of 180-mm radius rotates at the constant rate \( \omega_2 = 12 \text{ rad/s} \) with respect to arm \( CD \), which itself rotates at the constant rate \( \omega_1 = 8 \text{ rad/s} \) about the \( Y \) axis. Determine at the instant shown the velocity and acceleration of point \( B \) on the rim of the disk.
15.242 and 15.243 In the position shown the thin rod moves at a constant speed \( u = 3 \text{ in./s} \) out of the tube BC. At the same time tube BC rotates at the constant rate \( \omega_3 = 1.5 \text{ rad/s} \) with respect to arm CD. Knowing that the entire assembly rotates about the X axis at the constant rate \( \omega_1 = 1.2 \text{ rad/s} \), determine the velocity and acceleration of end A of the rod.

Fig. P15.242

Fig. P15.243

15.244 Two disks, each of 130-mm radius, are welded to the 500-mm rod CD. The rod-and-disks unit rotates at the constant rate \( \omega_2 = 3 \text{ rad/s} \) with respect to arm AB. Knowing that at the instant shown \( \omega_1 = 4 \text{ rad/s} \), determine the velocity and acceleration of (a) point E, (b) point F.

Fig. P15.244

15.245 In Prob. 15.244, determine the velocity and acceleration of (a) point G, (b) point H.

15.246 The vertical plate shown is welded to arm EFG, and the entire unit rotates at the constant rate \( \omega_3 = 1.6 \text{ rad/s} \) about the Y axis. At the same time, a continuous link belt moves around the perimeter of the plate at a constant speed \( u = 4.5 \text{ in./s} \). For the position shown, determine the acceleration of the link of the belt located (a) at point A, (b) at point B.

15.247 The vertical plate shown is welded to arm EFG, and the entire unit rotates at the constant rate \( \omega_3 = 1.6 \text{ rad/s} \) about the Y axis. At the same time, a continuous link belt moves around the perimeter of the plate at a constant speed \( u = 4.5 \text{ in./s} \). For the position shown, determine the acceleration of the link of the belt located (a) at point C, (b) at point D.
This chapter was devoted to the study of the kinematics of rigid bodies.

We first considered the translation of a rigid body [Sec. 15.2] and observed that in such a motion, all points of the body have the same velocity and the same acceleration at any given instant.

We next considered the rotation of a rigid body about a fixed axis [Sec. 15.3]. The position of the body is defined by the angle $\theta$ that the line $BP$, drawn from the axis of rotation to a point $P$ of the body, forms with a fixed plane (Fig. 15.39). We found that the magnitude of the velocity of $P$ is

$$v = \frac{ds}{dt} = r \dot{\theta} \sin \phi$$  \hfill (15.4)

where $\dot{\theta}$ is the time derivative of $\theta$. We then expressed the velocity of $P$ as

$$v = \frac{dr}{dt} = \omega \times r$$  \hfill (15.5)

where the vector

$$\omega = \omega k = \dot{\theta} k$$  \hfill (15.6)

is directed along the fixed axis of rotation and represents the angular velocity of the body.

Denoting by $\alpha$ the derivative $d\omega/dt$ of the angular velocity, we expressed the acceleration of $P$ as

$$a = \alpha \times r + \omega \times (\omega \times r)$$  \hfill (15.8)

Differentiating (15.6), and recalling that $k$ is constant in magnitude and direction, we found that

$$\alpha = \alpha k = \omega k = \ddot{\theta} k$$  \hfill (15.9)

The vector $\alpha$ represents the angular acceleration of the body and is directed along the fixed axis of rotation.
Next we considered the motion of a representative slab located in a plane perpendicular to the axis of rotation of the body (Fig. 15.40). Since the angular velocity is perpendicular to the slab, the velocity of a point \( P \) of the slab was expressed as 

\[ \mathbf{v} = \omega \mathbf{k} \times \mathbf{r} \]  
(15.10)

where \( \mathbf{v} \) is contained in the plane of the slab. Substituting \( \omega = \omega \mathbf{k} \) and \( \alpha = \alpha \mathbf{k} \) into (15.8), we found that the acceleration of \( P \) could be resolved into tangential and normal components (Fig. 15.41) respectively equal to

\[ \mathbf{a}_t = \alpha \mathbf{k} \times \mathbf{r} \quad a_t = r \alpha \]
\[ \mathbf{a}_n = -\omega^2 \mathbf{r} \quad a_n = r \omega^2 \]  
(15.11)

Recalling Eqs. (15.6) and (15.9), we obtained the following expressions for the angular velocity and the angular acceleration of the rotating slab [Sec. 15.4]:

\[ \omega = \frac{d\theta}{dt} \]  
(15.12)
\[ \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \]  
(15.13)

or

\[ \alpha = \omega \frac{d\omega}{d\theta} \]  
(15.14)

We noted that these expressions are similar to those obtained in Chap. 11 for the rectilinear motion of a particle.

Two particular cases of rotation are frequently encountered: uniform rotation and uniformly accelerated rotation. Problems involving either of these motions can be solved by using equations similar to those used in Secs. 11.4 and 11.5 for the uniform rectilinear motion and the uniformly accelerated rectilinear motion of a particle, but where \( x, v, \) and \( a \) are replaced by \( \theta, \omega, \) and \( \alpha \), respectively [Sample Prob. 15.1].
The most general plane motion of a rigid slab can be considered as the sum of a translation and a rotation [Sec. 15.5]. For example, the slab shown in Fig. 15.42 can be assumed to translate with point A, while simultaneously rotating about A. It follows [Sec. 15.6] that the velocity of any point B of the slab can be expressed as

\[ \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA} \quad (15.17) \]

where \(\mathbf{v}_A\) is the velocity of A and \(\mathbf{v}_{BA}\) the relative velocity of B with respect to A or, more precisely, with respect to axes \(x'y'\) translating with A. Denoting by \(\mathbf{r}_{BA}\) the position vector of B relative to A, we found that

\[ \mathbf{v}_{BA} = \hat{\mathbf{o}} \times \mathbf{r}_{BA} \quad \mathbf{v}_{BA} = r\omega \quad (15.18) \]

The fundamental equation (15.17) relating the absolute velocities of points A and B and the relative velocity of B with respect to A was expressed in the form of a vector diagram and used to solve problems involving the motion of various types of mechanisms [Sample Probs. 15.2 and 15.3].

Another approach to the solution of problems involving the velocities of the points of a rigid slab in plane motion was presented in Sec. 15.7 and used in Sample Probs. 15.4 and 15.5. It is based on the determination of the instantaneous center of rotation C of the slab (Fig. 15.43).
The fact that any plane motion of a rigid slab can be considered as the sum of a translation of the slab with a reference point and a rotation about this reference point was used in Sec. 15.8 to relate the absolute accelerations of any two points and of the slab and the relative acceleration of with respect to . We had

\[ \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \]  

(15.21)

where \( \mathbf{a}_{B/A} \) consisted of a normal component \( \mathbf{a}_{B/A}^n \) of magnitude \( r\alpha^2 \) directed toward and a tangential component \( \mathbf{a}_{B/A}^t \) of magnitude \( r\alpha \) perpendicular to the line AB (Fig. 15.44). The fundamental relation (15.21) was expressed in terms of vector diagrams or vector equations and used to determine the accelerations of given points of various mechanisms [Sample Probs. 15.6 through 15.8]. It should be noted that the instantaneous center of rotation C considered in Sec. 15.7 cannot be used for the determination of accelerations, since point C, in general, does not have zero acceleration.

In the case of certain mechanisms, it is possible to express the coordinates and of all significant points of the mechanism by means of simple analytic expressions containing a single parameter. The components of the absolute velocity and acceleration of a given point are then obtained by differentiating twice with respect to the time \( t \) the coordinates and of that point [Sec. 15.9].

While the rate of change of a vector is the same with respect to a fixed frame of reference and with respect to a frame in translation, the rate of change of a vector with respect to a rotating frame is different. Therefore, in order to study the motion of a particle relative to a rotating frame we first had to compare the rates of change of a general vector \( \mathbf{Q} \) with respect to a fixed frame \( OXYZ \) and with respect to a frame \( Oxyz \) rotating with an angular velocity \( \mathbf{\Omega} \) [Sec. 15.10] (Fig. 15.45). We obtained the fundamental relation

\[ (\dot{\mathbf{Q}})_{OXYZ} = (\dot{\mathbf{Q}})_{Oxyz} + \mathbf{\Omega} \times \mathbf{Q} \]  

(15.31)

and we concluded that the rate of change of the vector \( \mathbf{Q} \) with respect to the fixed frame \( OXYZ \) is made up of two parts: The first part represents the rate of change of \( \mathbf{Q} \) with respect to the rotating frame \( Oxyz \); the second part, \( \mathbf{\Omega} \times \mathbf{Q} \), is induced by the rotation of the frame \( Oxyz \).
The next part of the chapter [Sec. 15.11] was devoted to the two-dimensional kinematic analysis of a particle $P$ moving with respect to a frame $\mathcal{F}$ rotating with an angular velocity $\boldsymbol{\Omega}$ about a fixed axis (Fig. 15.46). We found that the absolute velocity of $P$ could be expressed as

$$v_P = v_{P'} + v_{P\mathcal{F}} \quad (15.33)$$

where $v_P$ = absolute velocity of particle $P$
$v_{P'}$ = velocity of point $P'$ of moving frame $\mathcal{F}$ coinciding with $P$
$v_{P\mathcal{F}}$ = velocity of $P$ relative to moving frame $\mathcal{F}$

We noted that the same expression for $v_P$ is obtained if the frame is in translation rather than in rotation. However, when the frame is in rotation, the expression for the acceleration of $P$ is found to contain an additional term $\mathbf{a}_c$ called the complementary acceleration or Coriolis acceleration. We wrote

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P\mathcal{F}} + \mathbf{a}_c \quad (15.36)$$

where $\mathbf{a}_P$ = absolute acceleration of particle $P$
$\mathbf{a}_{P'}$ = acceleration of point $P'$ of moving frame $\mathcal{F}$ coinciding with $P$
$\mathbf{a}_{P\mathcal{F}}$ = acceleration of $P$ relative to moving frame $\mathcal{F}$
$\mathbf{a}_c = 2\boldsymbol{\Omega} \times (\mathbf{r})_{O\mathcal{F}} = 2\boldsymbol{\Omega} \times v_{P\mathcal{F}}$
$\mathbf{a}_c$ = complementary, or Coriolis, acceleration

Since $\boldsymbol{\Omega}$ and $v_{P\mathcal{F}}$ are perpendicular to each other in the case of plane motion, the Coriolis acceleration was found to have a magnitude $a_c = 2\boldsymbol{\Omega} v_{P\mathcal{F}}$ and to point in the direction obtained by rotating the vector $v_{P\mathcal{F}}$ through $90^\circ$ in the sense of rotation of the moving frame. Formulas (15.33) and (15.36) can be used to analyze the motion of mechanisms which contain parts sliding on each other [Sample Probs. 15.9 and 15.10].

The last part of the chapter was devoted to the study of the kinematics of rigid bodies in three dimensions. We first considered the motion of a rigid body with a fixed point [Sec. 15.12]. After proving that the most general displacement of a rigid body with a fixed point $O$ is equivalent to a rotation of the body about an axis through $O$, we were able to define the angular velocity $\boldsymbol{\omega}$ and the instantaneous axis of rotation of the body at a given instant. The velocity of a point $P$ of the body (Fig. 15.47) could again be expressed as

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r} \quad (15.37)$$

Differentiating this expression, we also wrote

$$\mathbf{a} = \mathbf{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad (15.38)$$

However, since the direction of $\boldsymbol{\omega}$ changes from one instant to the next, the angular acceleration $\mathbf{\alpha}$ is, in general, not directed along the instantaneous axis of rotation [Sample Prob. 15.11].
It was shown in Sec. 15.13 that the most general motion of a rigid body in space is equivalent, at any given instant, to the sum of a translation and a rotation. Considering two particles $A$ and $B$ of the body, we found that

$$ v_B = v_A + v_{B/A} \quad (15.42) $$

where $v_{B/A}$ is the velocity of $B$ relative to a frame $AX'Y'Z'$ attached to $A$ and of fixed orientation (Fig. 15.48). Denoting by $r_{B/A}$ the position vector of $B$ relative to $A$, we wrote

$$ v_B = v_A + \omega \times r_{B/A} \quad (15.43) $$

where $\omega$ is the angular velocity of the body at the instant considered [Sample Prob. 15.12]. The acceleration of $B$ was obtained by a similar reasoning. We first wrote

$$ a_B = a_A + a_{B/A} $$

and, recalling Eq. (15.38),

$$ a_B = a_A + \alpha \times r_{B/A} + \omega \times (\omega \times r_{B/A}) \quad (15.44) $$

In the final two sections of the chapter we considered the three-dimensional motion of a particle $P$ relative to a frame $Oxyz$ rotating with an angular velocity $\Omega$ with respect to a fixed frame $OXYZ$ (Fig. 15.49). In Sec. 15.14 we expressed the absolute velocity $v_P$ of $P$ as

$$ v_P = v_{P'} + v_{P/\bar{F}} \quad (15.46) $$

where $v_P = $ absolute velocity of particle $P$

$v_{P'} = $ velocity of point $P'$ of moving frame $\bar{F}$ coinciding with $P$

$v_{P/\bar{F}} = $ velocity of $P$ relative to moving frame $\bar{F}$
The absolute acceleration \( \mathbf{a}_P \) of \( P \) was then expressed as
\[
\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c \tag{15.48}
\]
where \( \mathbf{a}_P = \) absolute acceleration of particle \( P \)
\( \mathbf{a}_{P'} = \) acceleration of point \( P' \) of moving frame \( \mathcal{F} \) coinciding with \( P \)
\( \mathbf{a}_{P/\mathcal{F}} = \) acceleration of \( P \) relative to moving frame \( \mathcal{F} \)
\( \mathbf{a}_c = 2\mathbf{\Omega} \times (\mathbf{v})_{Oxyz} = 2\mathbf{\Omega} \times \mathbf{v}_{P/\mathcal{F}} \)
= complementary, or Coriolis, acceleration

It was noted that the magnitude \( a_c \) of the Coriolis acceleration is not equal to \( 2\mathbf{\Omega} v_{P/\mathcal{F}} \) [Sample Prob. 15.13] except in the special case when \( \mathbf{\Omega} \) and \( \mathbf{v}_{P/\mathcal{F}} \) are perpendicular to each other.

We also observed [Sec. 15.15] that Eqs. (15.46) and (15.48) remain valid when the frame \( AXyz \) moves in a known, but arbitrary, fashion with respect to the fixed frame \( OXYZ \) (Fig. 15.50), provided that the motion of \( A \) is included in the terms \( \mathbf{v}_{P'} \) and \( \mathbf{a}_{P'} \) representing the absolute velocity and acceleration of the coinciding point \( P' \).

**Frame of reference in general motion**

Rotating frames of reference are particularly useful in the study of the three-dimensional motion of rigid bodies. Indeed, there are many cases where an appropriate choice of the rotating frame will lead to a simpler analysis of the motion of the rigid body than would be possible with axes of fixed orientation [Sample Probs. 15.14 and 15.15].

[Fig. 15.50]
15.248 Knowing that at the instant shown crank BC has a constant angular velocity of 45 rpm clockwise, determine the acceleration (a) of point A, (b) of point D.

15.249 The rotor of an electric motor has a speed of 1800 rpm when the power is cut off. The rotor is then observed to come to rest after executing 1550 revolutions. Assuming uniformly accelerated motion, determine (a) the angular acceleration of the rotor, (b) the time required for the rotor to come to rest.

15.250 A disk of 0.15-m radius rotates at the constant rate \( \omega_2 \) with respect to plate BC, which itself rotates at the constant rate \( \omega_1 \) about the y axis. Knowing that \( \omega_1 = \omega_2 = 3 \) rad/s, determine, for the position shown, the velocity and acceleration (a) of point D, (b) of point F.

15.251 The fan of an automobile engine rotates about a horizontal axis parallel to the direction of motion of the automobile. When viewed from the rear of the engine, the fan is observed to rotate clockwise at the rate of 2500 rpm. Knowing that the automobile is turning right along a path of radius 12 m at a constant speed of 12 km/h, determine the angular acceleration of the fan at the instant the automobile is moving due north.
15.252 A drum of radius 4.5 in. is mounted on a cylinder of radius 7.5 in. A cord is wound around the drum, and its extremity $E$ is pulled to the right with a constant velocity of 15 in./s, causing the cylinder to roll without sliding on plate $F$. Knowing that plate $F$ is stationary, determine (a) the velocity of the center of the cylinder, (b) the acceleration of point $D$ of the cylinder.

15.253 Solve Prob. 15.252, assuming that plate $F$ is moving to the right with a constant velocity of 9 in./s.

15.254 Water flows through a curved pipe $AB$ that rotates with a constant clockwise angular velocity of 90 rpm. If the velocity of the water relative to the pipe is 8 m/s, determine the total acceleration of a particle of water at point $P$.

15.255 Rod BC of length 24 in. is connected by ball-and-socket joints to a rotating arm $AB$ and to a collar $C$ that slides on the fixed rod $DE$. Knowing that the length of arm $AB$ is 4 in. and that it rotates at the constant rate $\omega_1 = 10$ rad/s, determine the velocity of collar $C$ when $\theta = 0$.

15.256 Solve Prob. 15.255, assuming that $\theta = 90^\circ$. 
15.257 Crank AB has a constant angular velocity of 1.5 rad/s counterclockwise. For the position shown, determine (a) the angular velocity of rod BD, (b) the velocity of collar D.

![Diagram of Crank AB and Rod BD](image)

**Fig. P15.257 and P15.258**

15.258 Crank AB has a constant angular velocity of 1.5 rad/s counterclockwise. For the position shown, determine (a) the angular acceleration of rod BD, (b) the acceleration of collar D.

15.259 Rod AB of length 125 mm is attached to a vertical rod that rotates about the y axis at the constant rate $\omega_1 = 5$ rad/s. Knowing that the angle formed by rod AB and the vertical is increasing at the constant rate $d\beta/dt = 3$ rad/s, determine the velocity and acceleration of end B of the rod when $\beta = 30^\circ$.

![Diagram of Rod AB attached to a vertical rod](image)

**Fig. P15.259**
15.C1 The disk shown has a constant angular velocity of 500 rpm counterclockwise. Knowing that rod $BD$ is 250 mm long, use computational software to determine and plot for values of $\theta$ from 0 to 360° and using 30° increments, the velocity of collar $D$ and the angular velocity of rod $BD$. Determine the two values of $\theta$ for which the speed of collar $D$ is zero.

Fig. P15.C1

15.C2 Two rotating rods are connected by a slider block $P$ as shown. Knowing that rod $BP$ rotates with a constant angular velocity of 6 rad/s counterclockwise, use computational software to determine and plot for values of $\theta$ from 0 to 180° the angular velocity and angular acceleration of rod $AE$. Determine the value of $\theta$ for which the angular acceleration $\alpha_{AE}$ of rod $AE$ is maximum and the corresponding value of $\alpha_{AE}$.

Fig. P15.C2
15.C3 In the engine system shown, \( l = 160 \) mm and \( b = 60 \) mm. Knowing that crank \( AB \) rotates with a constant angular velocity of 1000 rpm clockwise, use computational software to determine and plot for values of \( \theta \) from 0 to 180° and using 10° increments, (a) the angular velocity and angular acceleration of rod \( BD \), (b) the velocity and acceleration of the piston \( P \).

15.C4 Rod \( AB \) moves over a small wheel at \( C \) while end \( A \) moves to the right with a constant velocity of 180 mm/s. Use computational software to determine and plot for values of \( \theta \) from 20° to 90° and using 5° increments, the velocity of point \( B \) and the angular acceleration of the rod. Determine the value of \( \theta \) for which the angular acceleration \( \alpha \) of the rod is maximum and the corresponding value of \( \alpha \).

15.C5 Rod \( BC \) of length 24 in. is connected by ball-and-socket joints to the rotating arm \( AB \) and to collar \( C \) that slides on the fixed rod \( DE \). Arm \( AB \) of length 4 in. rotates in the \( XY \) plane with a constant angular velocity of 10 rad/s. Use computational software to determine and plot for values of \( \theta \) from 0 to 360° the velocity of collar \( C \). Determine the two values of \( \theta \) for which the velocity of collar \( C \) is zero.
15.C6 Rod $AB$ of length 25 in. is connected by ball-and-socket joints to collars $A$ and $B$, which slide along the two rods shown. Collar $B$ moves toward support $E$ at a constant speed of 20 in./s. Denoting by $d$ the distance from point $C$ to collar $B$, use computational software to determine and plot the velocity of collar $A$ for values of $d$ from 0 to 15 in.

**Fig. P15.C6**
Three-bladed wind turbines, similar to the ones shown in this picture of a wind farm, are currently the most common design. In this chapter you will learn to analyze the motion of a rigid body by considering the motion of its mass center, the motion relative to its mass center, and the external forces acting on it.